Quantum Computing

- Lectures 13 and 14 (June 25-26, 2025)

- Today:
  - Quantum Fourier Transformation
  - Phase Estimation

QFT

- Quantum Fourier Transformation

$$\mathbf{QFT_N}\colon |j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i j k}{N}} |k\rangle$$

Quantum Fourier Transformation

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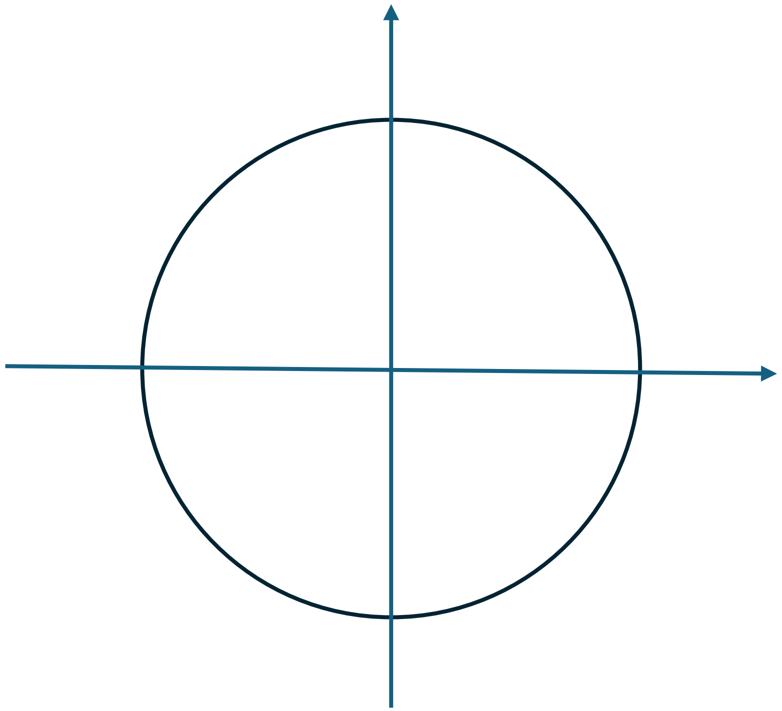

What is $e^{2\pi i (t)/N}$?

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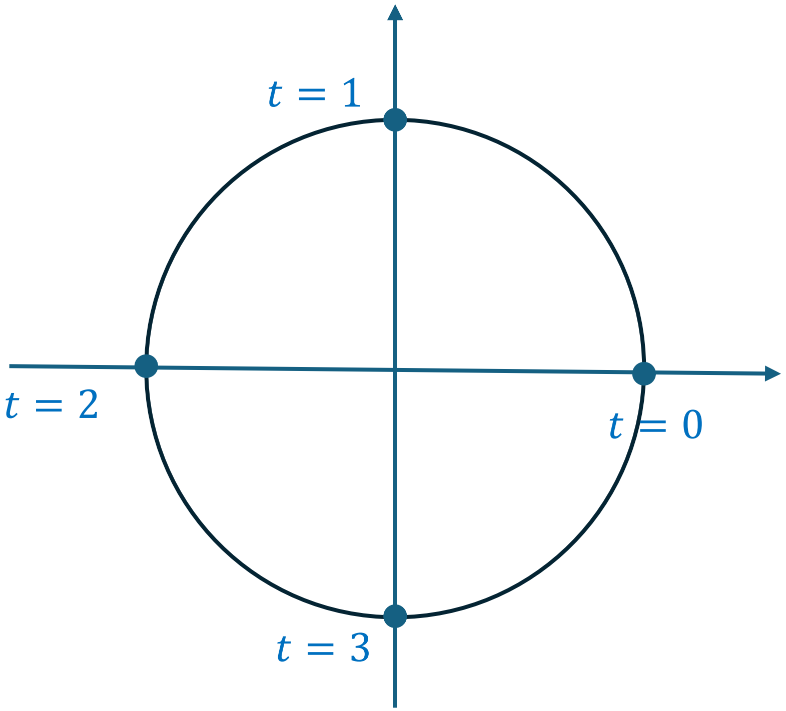
Quantum Fourier Transformation

E.g., $N = 4$



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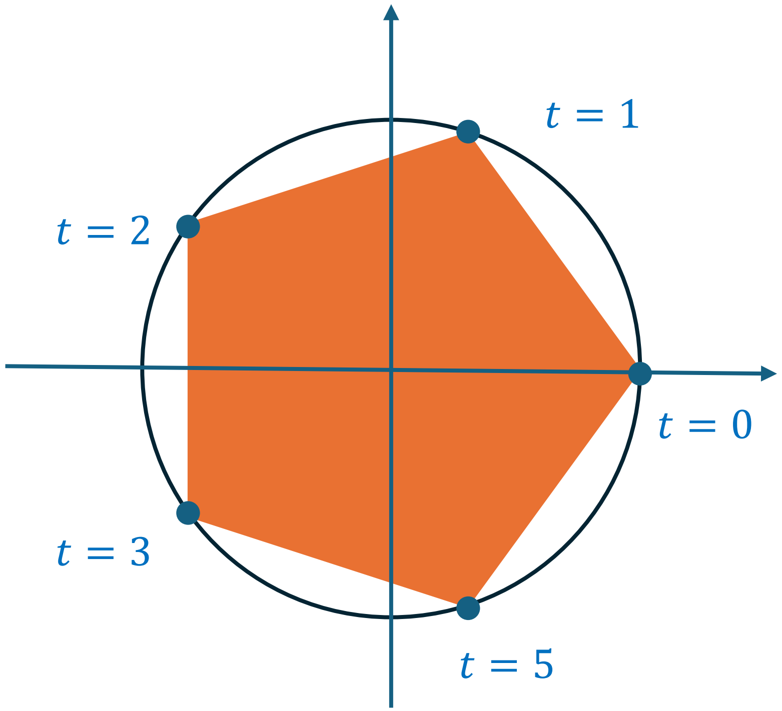
Quantum Fourier Transformation

E.g., $N = 5$



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QFT

$$\mathbf{QFT_N^\dagger}: |j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-\frac{2\pi ijk}{N}} |k\rangle$$

Inverse QFT

$$\mathbf{QFT_N^\dagger QFT_N} = \boldsymbol{I}$$

Inverse QFT

- Inverse Quantum Fourier Transformation

$$\mathbf{QFT}_{\mathbf{N}}^{\dagger}: |j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-\frac{2\pi i j k}{N}} |k\rangle$$

Inverse QFT

- Another way to understand inverse QFT:

$$|j\rangle \xrightarrow{\mathbf{QFT}_{\mathbf{N}}} \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2\pi i j k}{N}} |k\rangle \xrightarrow{\mathbf{QFT}_{\mathbf{N}}^{\dagger}} |j\rangle$$

Inverse QFT

- Inverse Quantum Fourier Transformation

$$\mathbf{QFT}_{\mathbf{N}}^{\dagger}\colon \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{\frac{2\pi i \boldsymbol{j} k}{N}}|k\rangle \mapsto |j\rangle$$

Inverse QFT

- Extract ***j*** from the phases!

Inverse QFT

- Inverse Quantum Fourier Transformation

$$\mathbf{QFT}_{\boldsymbol{n}}^{\dagger}\colon \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i k\left(\frac{j}{2^n}\right)} |k\rangle \mapsto |j\rangle$$

Inverse QFT

- Extract j from the phases!
- Let $N = 2^n, j \in \{0, 1, 2, \dots, 2^n - 1\}$
- How can we relate $\frac{j}{2^n}$ to $|j\rangle$?

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- How can we relate $\frac{\boldsymbol{j}}{\boldsymbol{2^n}}$ to $|j\rangle$?
- **Observation:** $\boldsymbol{j} = j_1 \cdot 2^{n-1} + j_2 \cdot 2^{n-2} + \cdots + j_{n-1} \cdot 2^1 + j_n \cdot 1$

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Inverse QFT

- $\frac{j}{2^n} = j_1 \cdot 2^{-1} + j_2 \cdot 2^{-2} + \cdots + j_{n-1} \cdot 2^{-(n-1)} + j_n \cdot 2^{-n}$
- **Fact:** $e^{2\pi i k\left(a+\frac{j}{2^n}\right)} = e^{2\pi i k\left(\frac{j}{2^n}\right)}$ for any integer $a \geq 1$ **(always mod 1 on the exponent)**

Inverse QFT

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- Let $0.j_1 j_2 j_3 \ldots j_l = j_1 \cdot 2^{-1} + j_2 \cdot 2^{-2} + \cdots + j_{l-1} \cdot 2^{-(l-1)} + j_l \cdot 2^{-l}$
- **Fact:** $(0.j_1 j_2 j_3 \ldots j_l) \cdot 2^m = 0.j_{m+1} \ldots j_{l-m}$

Inverse QFT

- These notations give us an alternative way to understand (Inverse) QFT…

$$\mathbf{QFT}_{\boldsymbol{n}}: |j\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i k (0.j_1 j_2 \dots j_n)} |k\rangle$$

QFT

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Inverse QFT

- Claim: (Leave as an exercise tomorrow...)

$$\frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i k (0.j_1 j_2 \ldots j_n)} |k\rangle = \frac{1}{\sqrt{2^n}} \left(\begin{array}{c} |0\rangle + e^{2\pi i (0.j_n)} |1\rangle \\ \otimes |0\rangle + e^{2\pi i (0.j_{n-1} j_n)} |1\rangle \\ \otimes |0\rangle + e^{2\pi i (0.j_{n-2} j_{n-1} j_n)} |1\rangle \\ \vdots \\ \otimes |0\rangle + e^{2\pi i (0.j_1 j_2 j_3 \ldots j_n)} |1\rangle \end{array} \right)$$

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Inverse QFT

- Applications: **Phase Estimation**

Phase Estimation

- Let $\boldsymbol{U}$ be a unitary and $|u\rangle$ be an eigenvector of $\boldsymbol{U}$, i.e., $\boldsymbol{U}|u\rangle = \lambda|u\rangle,\ \lambda \in \mathbb{C}$
- By the normalized condition: $|\lambda| = 1 \Rightarrow \boldsymbol{\lambda} = \boldsymbol{e}^{\boldsymbol{2\pi i \varphi}}$ for some $\boldsymbol{\varphi} \in [0,1)$ (Quick question: Why?)
- $\boldsymbol{U}|u\rangle = \boldsymbol{e}^{\boldsymbol{2\pi i \varphi}}|u\rangle$
- By the notation introduced before: $\boldsymbol{U}|u\rangle = \boldsymbol{e}^{\boldsymbol{2\pi i \varphi}}|u\rangle = \boldsymbol{e}^{\boldsymbol{2\pi i}(0.\varphi_1\varphi_2\varphi_3\ldots)}|u\rangle$

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- **Goal of Phase Estimation: Compute or Estimate** $\boldsymbol{\varphi} = 0.\varphi_1\varphi_2\varphi_3 \ldots$

- What does estimation mean? Compute $\boldsymbol{\varphi}' = 0.\varphi_1\varphi_2\varphi_3 \ldots \varphi_n$ so that $|\boldsymbol{\varphi} - \boldsymbol{\varphi}'|$ is small

Phase Estimation

- Phase estimation via inverse QFT

$$\mathbf{QFT}_{\boldsymbol{n}}^{\dagger}\colon \frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k(0.j_1 j_2 \dots j_n)}|k\rangle \mapsto |j\rangle$$

Inverse QFT

- For $\boldsymbol{U}|u\rangle = e^{2\pi i\varphi}|u\rangle = e^{2\pi i(0.\varphi_1\varphi_2\varphi_3\dots)}|u\rangle$, if we have: $\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k(0.\varphi_1\varphi_2\varphi_3\dots)}|k\rangle$
- Then what is $\mathbf{QFT}_{\boldsymbol{n}}^{\dagger}\left(\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k(0.\varphi_1\varphi_2\varphi_3\dots)}|k\rangle\right)$?

Phase Estimation

- Phase estimation via inverse QFT
- For $\boldsymbol{U}|u\rangle = e^{2\pi i\varphi}|u\rangle = e^{2\pi i(0.\varphi_1\varphi_2\varphi_3\ldots)}|u\rangle$, suppose that we have: $\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k(0.\varphi_1\varphi_2\varphi_3\ldots)}|k\rangle$
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- **Case 1:** $\varphi = 0.\varphi_1\varphi_2 \ldots \varphi_t$ where $t \leq n$

$$\mathbf{QFT}_{\boldsymbol{n}}^{\dagger}\left(\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k(0.\varphi_1\varphi_2\ldots\varphi_t)}|k\rangle\right) \mapsto |\varphi_1\varphi_2 \ldots \varphi_t\varphi_{t+1} \ldots \varphi_n\rangle$$

By Inverse QFT ($\varphi_{t+1} \ldots \varphi_n = 0 \ldots 0$)

Phase Estimation

- Phase estimation via inverse QFT
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By Inverse QFT

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By Inverse QFT

Theorem (informal):

$$\Pr[|\varphi - \varphi'| \le 2^{-n+2}] \ge \frac{1}{2},$$

which means that φ' gives a good estimation of φ.

Phase Estimation

- Phase estimation via inverse QFT
- For $\boldsymbol{U}|u\rangle = e^{2\pi i\varphi}|u\rangle = e^{2\pi i(0.\varphi_1\varphi_2\varphi_3\dots)}|u\rangle$.
- Suppose that we have: $\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi ik(0.\varphi_1\varphi_2\varphi_3\dots)}|k\rangle$
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- **Answer:** $\mathbf{QFT}_{\boldsymbol{n}}^{\dagger}\left(\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi ik(0.\varphi_1\varphi_2\varphi_3\dots)}|k\rangle\right) \mapsto |\varphi'\rangle$, where φ' is a good estimation of φ

Phase Estimation

- inverse QFT for phase estimation

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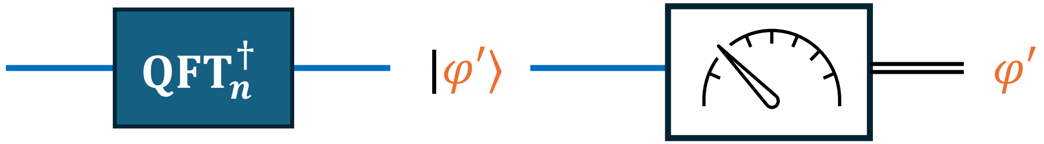

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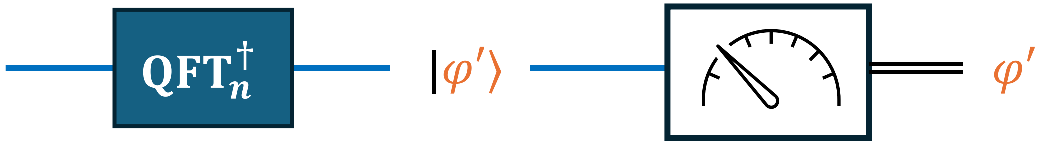

- How can we generate this state if we have the unitary and the eigenvector:

Given $\boldsymbol{U}$ and $|u\rangle$, generate $$\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k\varphi}|k\rangle$$

- (Leave as an exercise tomorrow)

Period Finding

- Suppose that we have a function f with a period $r < 2^L$.
- Namely, there exists a minimal $r > 0$ such that $f(x + r) = f(x)$
- Goal: Find r

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- Suppose we have the following state: (Let n be a large enough integer, e.g., $n \approx L + 2$)

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- We should have $\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle |f(x)\rangle = \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} (\dots)$

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We need a good basis to express this "periodic state"...

Period Finding

- Suppose that we have a function f with a period r
  - ...and $f(x_1) \neq f(x_2)$ for any distinct $x_1, x_2 \in \{0, \dots, r-1\}$.
- We define the following Fourier basis $\{|\hat{f}(0)\rangle, |\hat{f}(1)\rangle, \dots, |\hat{f}(r-1)\rangle\}$, where:

$$|\hat{f}(l)\rangle = \frac{1}{\sqrt{r}} \sum_{x=0}^{r-1} e^{-2\pi i \cdot l \cdot (\frac{x}{r})} |f(x)\rangle$$

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Some insights:
By this definition,
$|\hat{f}(l_1)\rangle$ is always orthogonal to $|\hat{f}(l_2)\rangle$ if $l_1 \neq l_2$

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- Then we also have:

$$|f(x)\rangle = \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} e^{2\pi i \cdot x \cdot \left(\frac{l}{r}\right)} |\hat{f}(l)\rangle$$

- Exercise (tomorrow):
  - Prove the states defined above constitute an **orthonormal basis.**
  - Prove the second equality.

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- And we also have: $|f(x)\rangle = \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} e^{2\pi i \cdot x \cdot \left(\frac{l}{r}\right)} \left|\hat{f}(l)\right\rangle$
- Continue the calculation and apply inverse QFT: $\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle |f(x)\rangle = \cdots = \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} \left|\left(\frac{l}{r}\right)'\right\rangle \left|\hat{f}(l)\right\rangle$

Period Finding

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- And we also have: $|f(x)\rangle = \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} e^{2\pi i \cdot x \cdot (\frac{l}{r})} \left|\hat{f}(l)\right\rangle$
- Continue the calculation and apply inverse QFT: $\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle |f(x)\rangle = \cdots = \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} \left|\left(\frac{l}{r}\right)'\right\rangle \left|\hat{f}(l)\right\rangle$
- Measure the first n-qubit system gives us a good estimation of $\left(\frac{l}{r}\right)$
- Apply many times, we get $\left\{\left(\frac{l_1}{r}\right)', \left(\frac{l_2}{r}\right)', \dots\right\}$, which allows us to recover r

Exercise

- (1) Prove this equality:

$$\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k(0.j_1 j_2 \dots j_n)}|k\rangle = \frac{1}{\sqrt{2^n}}\begin{pmatrix} \left(|0\rangle + e^{2\pi i(0.j_n)}|1\rangle\right) \\ \otimes\left(|0\rangle + e^{2\pi i(0.j_{n-1}j_n)}|1\rangle\right) \\ \otimes\left(|0\rangle + e^{2\pi i(0.j_{n-2}j_{n-1}j_n)}|1\rangle\right) \\ \vdots \\ \otimes\left(|0\rangle + e^{2\pi i(0.j_1 j_2 j_3 \dots j_n)}|1\rangle\right)\end{pmatrix}$$

- (2) Given $\boldsymbol{U}$ and $|u\rangle$ where $\boldsymbol{U}|u\rangle = e^{2\pi i k\varphi}|u\rangle$, generate

$$\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k\varphi}|k\rangle$$

- (3) Prove $|\hat{f}(l)\rangle = \frac{1}{\sqrt{r}}\sum_{x=0}^{r-1} e^{-2\pi i\cdot l\cdot(\frac{x}{r})}|f(x)\rangle$ forms a basis, and $|f(x)\rangle = \frac{1}{\sqrt{r}}\sum_{l=0}^{r-1} e^{2\pi i\cdot x\cdot(\frac{l}{r})}|\hat{f}(l)\rangle$
  - where r is the period of f
  - Suppose that $f(x_1) \neq f(x_2)$ for distinct $x_1, x_2 \in \{0, \dots, r-1\}$
- (4) How can we create the state $\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle|f(x)\rangle$ if we have $\boldsymbol{U}_f$?

Exercise

- (1) Prove this equality:

$$\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k(0.j_1 j_2 \dots j_n)}|k\rangle = \frac{1}{\sqrt{2^n}}\begin{pmatrix} (|0\rangle + e^{2\pi i(0.j_n)}|1\rangle) \\ \otimes (|0\rangle + e^{2\pi i(0.j_{n-1}j_n)}|1\rangle) \\ \otimes (|0\rangle + e^{2\pi i(0.j_{n-2}j_{n-1}j_n)}|1\rangle) \\ \vdots \\ \otimes (|0\rangle + e^{2\pi i(0.j_1 j_2 j_3 \dots j_n)}|1\rangle) \end{pmatrix}$$

Hint:

$$= \frac{1}{\sqrt{2^n}}\left(\otimes_{i=1}^{n}\left(\sum_{k_i=0}^{1} e^{2\pi i k_i(0.j_{n-i+1}\dots j_n)}|k_i\rangle\right)\right)$$

- (2) Given $\boldsymbol{U}$ and $|u\rangle$ where $\boldsymbol{U}|u\rangle = e^{2\pi i k\boldsymbol{\varphi}}|u\rangle$. Suppose that $\boldsymbol{\varphi} = 0.\varphi_1\varphi_2 \dots \varphi_n$. Generate

$$\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1} e^{2\pi i k\boldsymbol{\varphi}}|k\rangle$$

(Hint: Use (1) and controlled $\boldsymbol{U}^{2^j}$)

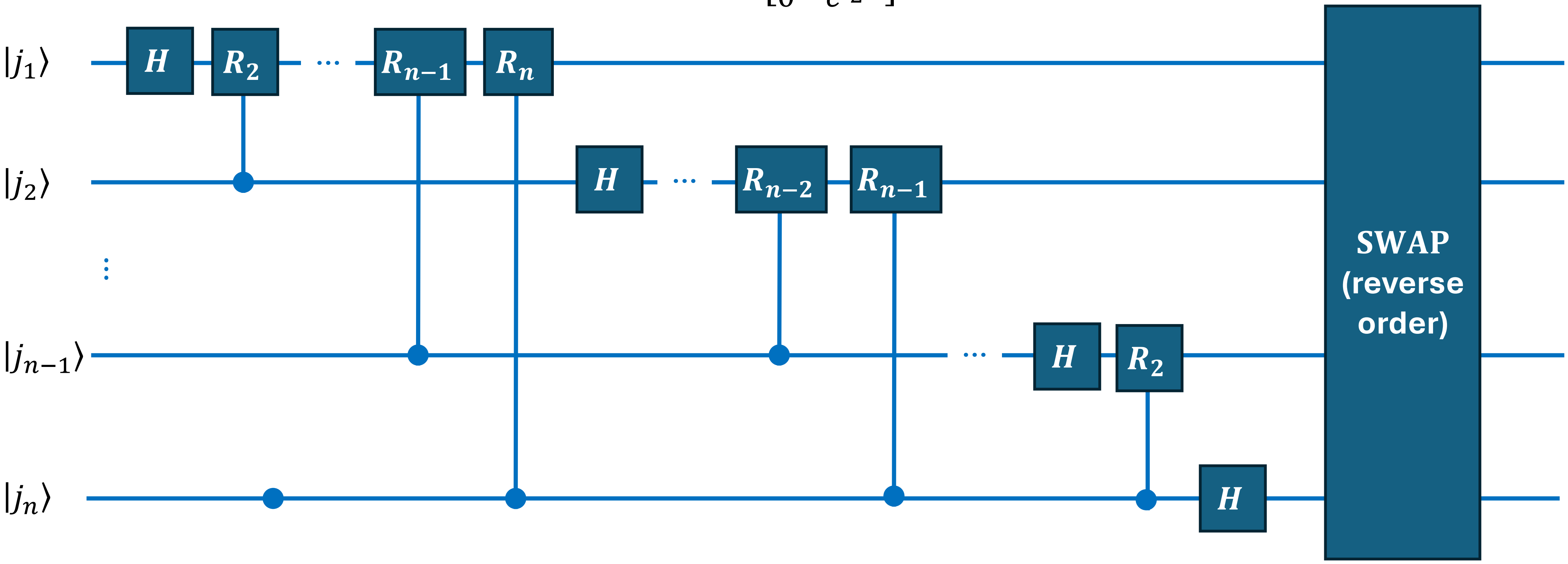
- (3) Prove $|\hat{f}(l)\rangle = \frac{1}{\sqrt{r}}\sum_{x=0}^{r-1} e^{-2\pi i \cdot l \cdot (\frac{x}{r})}|f(x)\rangle$ forms a basis, and $|f(x)\rangle = \frac{1}{\sqrt{r}}\sum_{l=0}^{r-1} e^{2\pi i \cdot x \cdot (\frac{l}{r})}|\hat{f}(l)\rangle$
    - where r is the period of f (Hint: $\sum_{l=0}^{r-1} e^{2\pi i \cdot l \cdot (\frac{a-b}{r})} = r$ if $a = b\ (mod\ r)$; Otherwise, $= 0$)
    - Suppose that $f(x_1) \neq f(x_2)$ for distinct $x_1, x_2 \in \{0, \dots, r-1\}$
- (4) How can we create the state $\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle|f(x)\rangle$ if we have $\boldsymbol{U}_f$?

Summary of QFT and inverse QFT

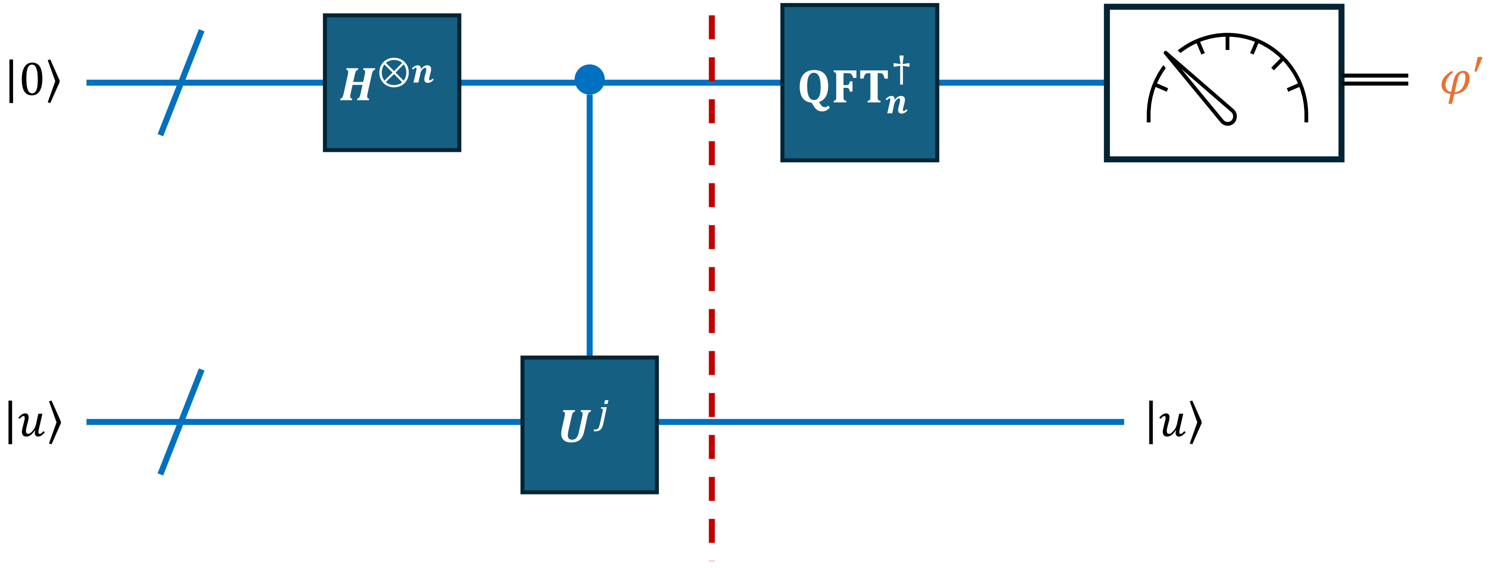
- Circuit for QFT (and similarly, inverse QFT)

$$R_k := \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{bmatrix}$$

Quick question:
How can we implement R_k?

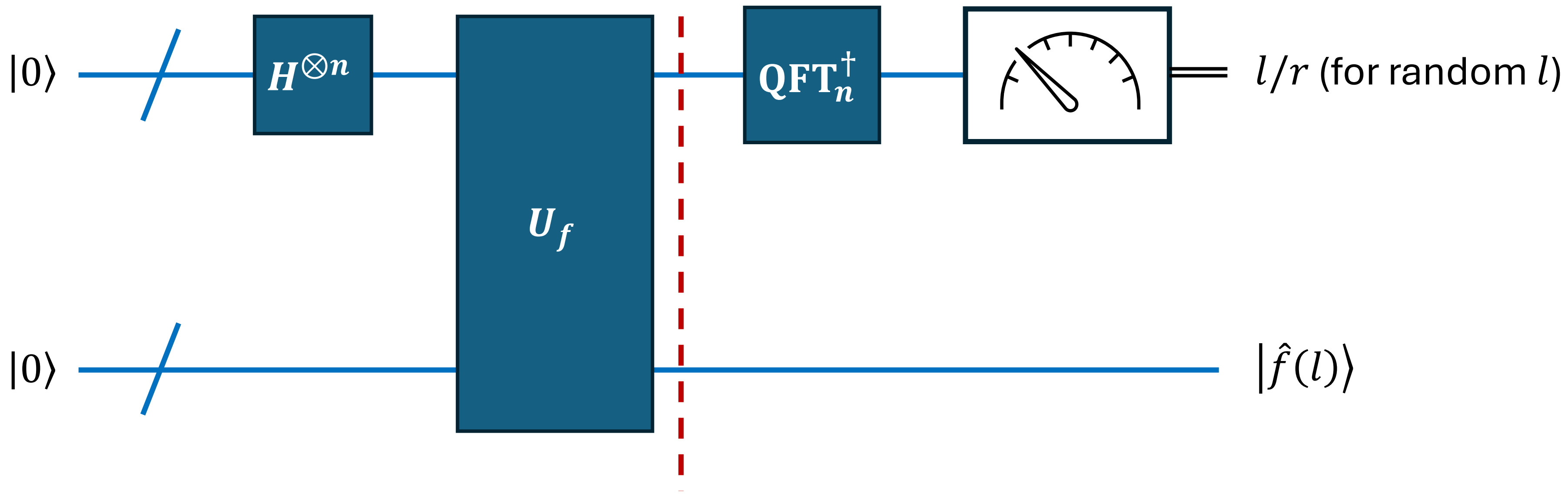


Summary of Phase estimation



$$\frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{2\pi i k \varphi} |k\rangle$$

Summary of Period Finding



$$\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle|f(x)\rangle$$

Next Week

- Order finding and Factoring
- **Shor's algorithm**

Reference

- **[NC00]:** Chapter 5
- **[KLM07]:** Chapter 7