Quantum Computing

- Week 13 (July 16-17, 2025)

- Topics:
  - A summary of quantum algorithms
  - Pure states and mixed states
  - Density operator and trace
  - Partial trace and partial measurement
  - Reduced density operator

Quantum Algorithms

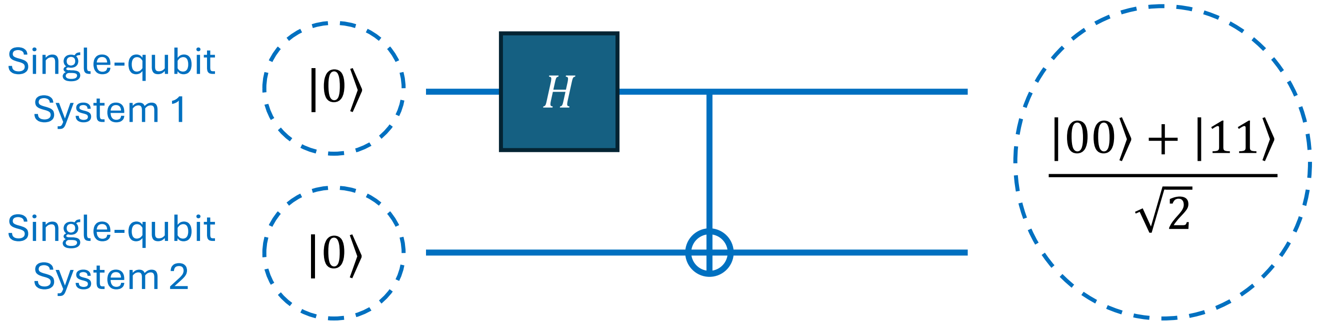
- Quantum algorithms that we have studied so far:
  - Deutsch-Jozsa algorithm
  - Superdense coding
  - Quantum teleportation
  - Quantum Fourier transformation and order finding
  - Grover search algorithm

Quantum Algorithms

Algorithm	Addressing problem	Classical "Complexity"	Quantum "Complexity"	Improvement
Deutsch-Jozsa	Balance functions	$O(2^n)$	$O(1)$	Exponentially
Superdense coding	Transmit classical info	-	1 qubit = 2-bit info (via 1 entangled pair)	-
Quantum teleportation	Teleport quantum states	2-bit info = 1 qubit (via 1 entangled pair)	-	-
Order/Period Finding (QFT)	Factoring, Discrete log	$2^{o(n)} \sim O(2^{n/2})$	$O(n^2)$ or $O(n \log n)$	Exponentially
Grover	Unstructured search	$O(2^n)$	$O\left(2^{n/2}\right)$	Quadratically

Mixed States, Recaps

- **Pure state:** Can be described by a state vector
- **Mixed state:** Cannot ...



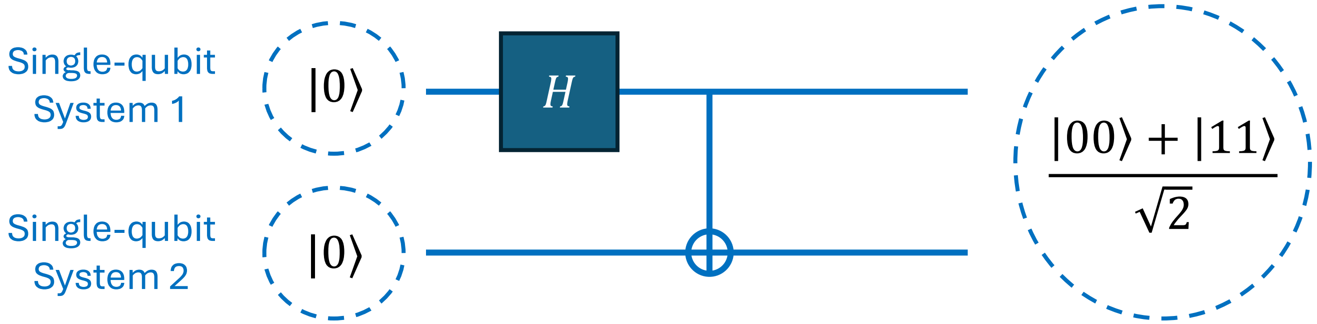
Small Exercise: (pure or mixed)

1. The initial state of system 1 is ____.
2. The states of systems 1 and 2 (after H and CNOT) are both ____.
3. The state of the total system (after H and CNOT) is ____.

Mixed States, Recaps

- **Pure state:** Can be described by a state vector
- **Mixed state:** Cannot …

How can we describe mixed states?



Small Exercise: (pure or mixed)

1. The initial state of system 1 is pure.
2. The states of systems 1 and 2 (after H and CNOT) are both mixed.
3. The state of the total system (after H and CNOT) is pure.

Density Operator

- Let $\{|\psi_i\rangle\}_i$ be a set of pure states, where i is an index
- Suppose that a quantum system is in $|\psi_i\rangle$ with probability p_i (s.t. $\sum_i p_i = 1$)
- Then we write the **density operator** $\boldsymbol{\rho}$ of the system as

$$\boldsymbol{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$$

- Examples:
  - A single-qubit system with state $|0\rangle$
  - A single-qubit system with state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$
  - A single-qubit system that is in state $|0\rangle$ with probability $\frac{1}{2}$ and in state $|1\rangle$ with probability $\frac{1}{2}$

Density Operator

- $\boldsymbol{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$ (where $\sum_i p_i = 1$)

What's the difference between the states of the two systems?

System 1: Sample two bits $b_1 b_2$ uniformly at random, and set its state as $|b_1 b_2\rangle$

System 2: $\frac{|00\rangle+|01\rangle+|10\rangle+|11\rangle}{2}$

Density Operator

- $\boldsymbol{\rho} = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$ (where $\sum_i p_i = 1$)

What's the difference between the states of the two systems?

System 1: Sample two bits $b_1 b_2$ uniformly at random, and set its state as $|b_1 b_2\rangle$

System 2: $\frac{|00\rangle+|01\rangle+|10\rangle+|11\rangle}{2}$

$$\boldsymbol{\rho}_1 = \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{\rho}_2 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Density Operator

- $\boldsymbol{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ (where $\sum_i p_i = 1$)
- Density operator **provides another way to formulate postulates** of quantum computing:

Postulate	State vector	Density operator
Describing the state of a system	$|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle$	$\boldsymbol{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
Unitary transformation	$|\psi\rangle \mapsto \boldsymbol{U}|\psi\rangle$	$\boldsymbol{\rho} \mapsto \boldsymbol{U}\boldsymbol{\rho}\boldsymbol{U}^\dagger$
Quantum measurement $\{M_m\}_m$	$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$, $|\psi_m\rangle = \frac{M_m}{\sqrt{p(m)}}|\psi\rangle$	$p(m) = \mathbf{tr}(M_m^\dagger M_m \boldsymbol{\rho})$, $\boldsymbol{\rho}_m = \frac{M_m \boldsymbol{\rho} M_m^\dagger}{p(m)}$
Composite system	$|\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle$	$|\boldsymbol{\rho}_1\rangle \otimes \cdots \otimes |\boldsymbol{\rho}_n\rangle$

Trace

- The **sum of diagonal elements:**

$$\boldsymbol{tr}(M) = \sum_i M_{ii}$$

- Properties of trace:
  - **Linearity:** $\boldsymbol{tr}(A + B) = \boldsymbol{tr}(A) + \boldsymbol{tr}(B), \boldsymbol{tr}(z \cdot A) = z \cdot \boldsymbol{tr}(A)$
  - **Cyclicity:** $\boldsymbol{tr}(AB) = \boldsymbol{tr}(BA)$ (similarly, $\boldsymbol{tr}(ABC) = \boldsymbol{tr}(CAB) = \boldsymbol{tr}(BCA)\ldots$)

- Several facts implied by **Cyclicity:**
  - $\boldsymbol{tr}(\boldsymbol{U\rho U^\dagger}) = \boldsymbol{tr}(\boldsymbol{\rho})$
  - $\boldsymbol{tr}(\boldsymbol{M}|\psi\rangle\langle\psi|) = \boldsymbol{tr}(\langle\psi|\boldsymbol{M}|\psi\rangle) = \langle\psi|\boldsymbol{M}|\psi\rangle$

Trace

- The **sum of diagonal elements:** $\boldsymbol{tr}(M) = \sum_i M_{ii}$

- Properties of trace:
  - **Linearity:** $\boldsymbol{tr}(A + B) = \boldsymbol{tr}(A) + \boldsymbol{tr}(B)$, $\boldsymbol{tr}(z \cdot A) = z \cdot \boldsymbol{tr}(A)$
  - **Cyclicity:** $\boldsymbol{tr}(AB) = \boldsymbol{tr}(BA)$ (similarly, $\boldsymbol{tr}(ABC) = \boldsymbol{tr}(CAB) = \boldsymbol{tr}(BCA) \ldots$)
- Several facts implied by **Cyclicity:**
  - $\boldsymbol{tr}(\boldsymbol{U\rho U^\dagger}) = \boldsymbol{tr}(\boldsymbol{\rho})$
  - $\boldsymbol{tr}(\boldsymbol{M}|\psi\rangle\langle\psi|) = \boldsymbol{tr}(\langle\psi|\boldsymbol{M}|\psi\rangle) = \langle\psi|\boldsymbol{M}|\psi\rangle$, (and thus $\boldsymbol{tr}(|\psi\rangle\langle\psi|) = \langle\psi|\psi\rangle = 1$)

- Some examples:
  - Unitary transformation formulated by state vectors v.s. by density operators
  - Measurement formulated by state vectors v.s. by density operators

Trace

- **Criterion** to decide if a state is mixed or pure: Let $\boldsymbol{\rho}$ be the density operator of a quantum system.
  - $\boldsymbol{tr}(\boldsymbol{\rho}^2) < 1$: Mixed state
  - $\boldsymbol{tr}(\boldsymbol{\rho}^2) = 1$: Pure state

Partial Trace and Reduced Density Operator

- Let:
  - A and B be two quantum systems
  - Let $|a_1\rangle$ and $|a_2\rangle$ be any two state vectors defined over (the state space of) A
  - Let $|b_1\rangle$ and $|b_2\rangle$ be any two state vectors defined over B
- **Partial Trace:**

$$\boldsymbol{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \coloneqq |a_1\rangle\langle a_2| \cdot \boldsymbol{tr}(|b_1\rangle\langle b_2|)$$

- Remark:
  1. $\boldsymbol{tr}$ maps an operator onto a complex number, but $\boldsymbol{tr}_B$ maps an operator onto an operator.
  2. The operator from $\boldsymbol{tr}_B$ is an operator defined over A

Partial Trace and Reduced Density Operator

- Let:
  - A and B be two quantum systems
  - Let $\boldsymbol{\rho}_{AB}$ be the state of the state space of the composite system $A \otimes B$

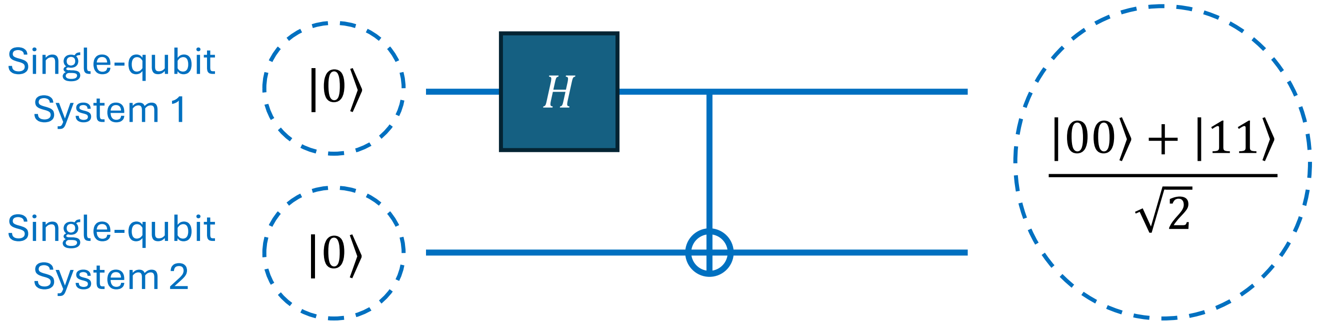
- **Reduced Density Operator** of A **:**

$$\boldsymbol{\rho}_A \coloneqq \boldsymbol{tr}_B(\boldsymbol{\rho}_{AB})$$

- Examples (of calculating $\boldsymbol{\rho}_A$) :
  1. $\boldsymbol{\rho}_{AB} = \rho \otimes \sigma$, where ρ is defined over A and σ is defined over B
  2. $\boldsymbol{\rho}_{AB} = |00\rangle\langle 00| (= |0\rangle\langle 0| \otimes |0\rangle\langle 0|)$
  3. $\boldsymbol{\rho}_{AB} = \left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right)\left(\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right) \otimes \left(\frac{\langle 0|-\langle 1|}{\sqrt{2}}\right)\right)$

Partial Trace and Reduced Density Operator

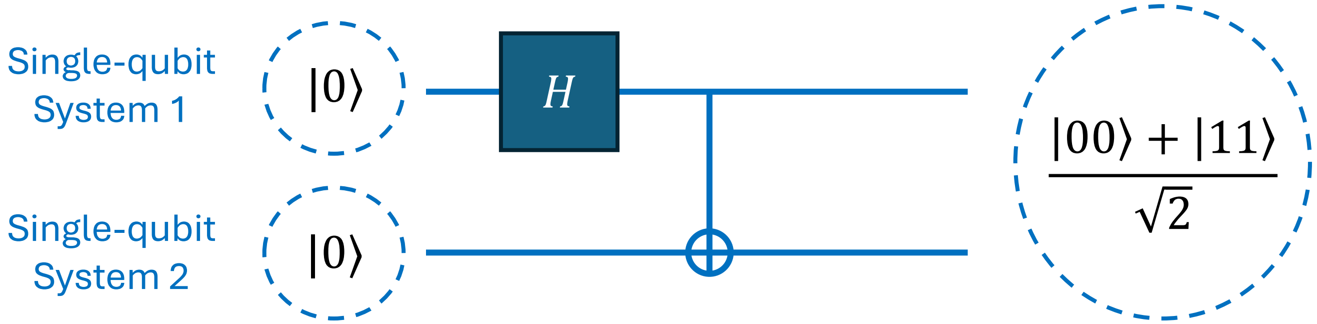
- The final state is $\boldsymbol{\rho}_{12} = \left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)$



- $\boldsymbol{\rho}_1 = \boldsymbol{tr}_2(\boldsymbol{\rho}_{12}) = ?$

Partial Trace and Reduced Density Operator

- The final state is $\boldsymbol{\rho}_{12} = \left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|00\rangle+|11\rangle}{\sqrt{2}}\right)$



- $\boldsymbol{\rho}_1 = \boldsymbol{tr}_2(\boldsymbol{\rho}_{12}) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

Pure state and Mixed state

- $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

- $\boldsymbol{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

Pure state and Mixed state

- $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$
  - $\boldsymbol{tr}(\boldsymbol{\rho}) = 1 \Rightarrow$ A pure state.
  - We "somehow" know the state of the system with **certainty** (i.e., the state is $|\psi\rangle$ with probability 1)
  - More generally, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we still know the state of the system with **certainty** (though we do not know α and β, the system is still **in a definite pure state…**)

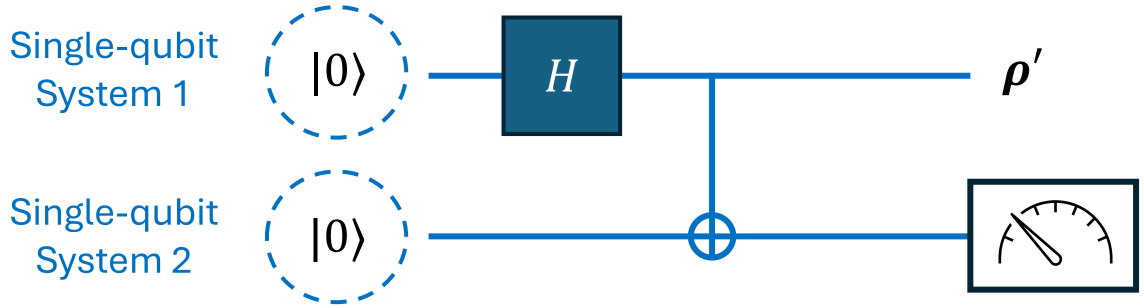
- $\boldsymbol{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

Pure state and Mixed state

- $\boldsymbol{\rho} = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$
  - $\boldsymbol{tr}(\boldsymbol{\rho}) = 1 \Rightarrow$ A pure state.
  - We “somehow” know the state of the system with **certainty** (i.e., the state is $|\psi\rangle$ with probability 1)
  - More generally, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we still know the state of the system with **certainty** (though we do not know α and β, the system is still **in a definite pure state...**)

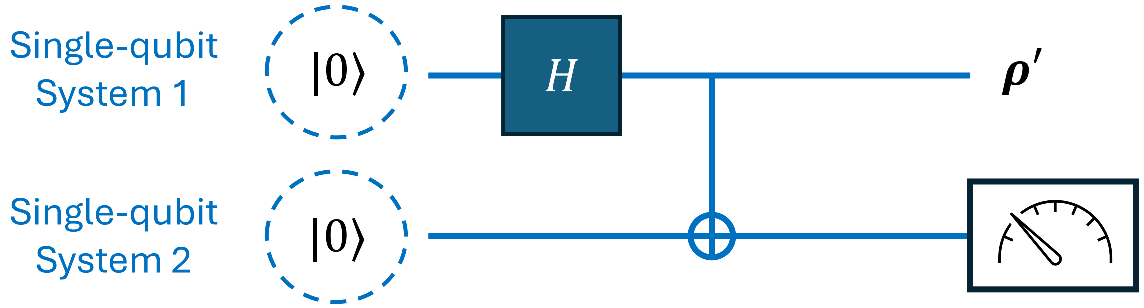
- $\boldsymbol{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
  - $\boldsymbol{tr}(\boldsymbol{\rho}) = \frac{1}{2} < 1 \Rightarrow$ A mixed state
  - We are **uncertain** about the exact state the system is in...

Pure state and Mixed state



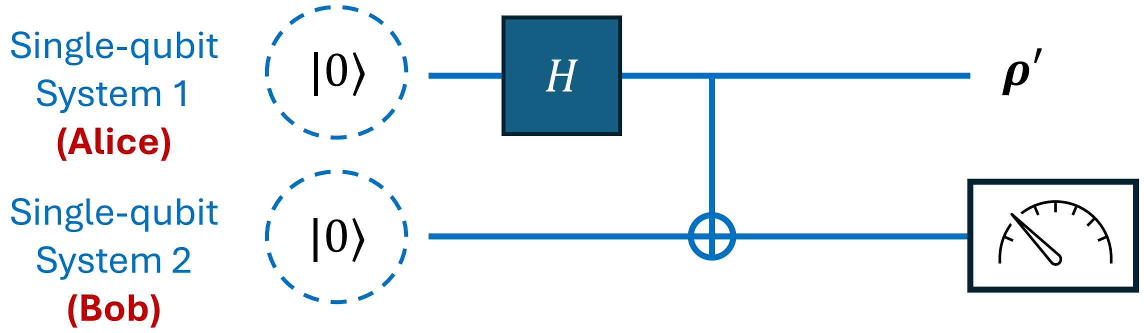
- $\rho' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
  - $tr(\rho) = \frac{1}{2} < 1 \Rightarrow$ A mixed state
  - We are **uncertain** about the exact state the system is in...

Pure state and Mixed state



- $\rho' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
  - $tr(\rho) = \frac{1}{2} < 1 \Rightarrow$ A mixed state
  - We are **uncertain** about the exact state the system is in...
  - The measurement (on system 2) outcome is b with probability $\frac{1}{2}$, corresponding to the state and the probability distribution of system 1...

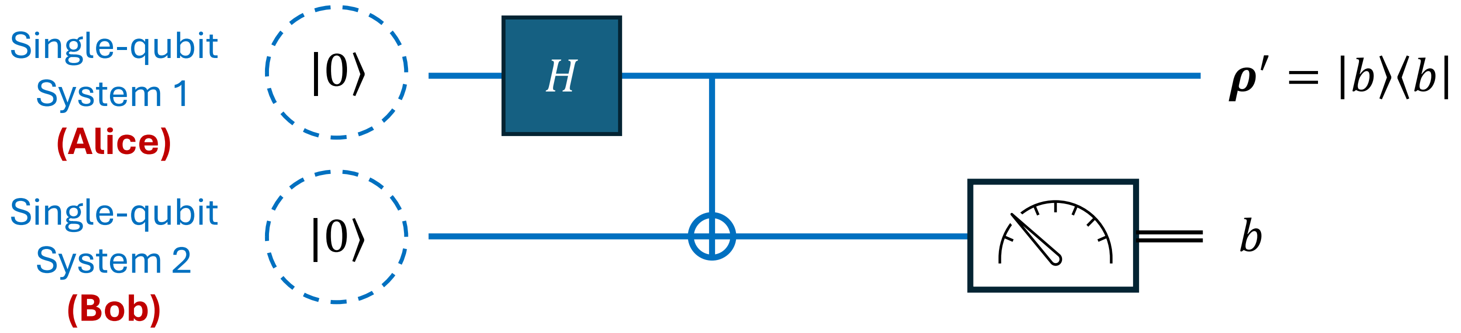
Pure state and Mixed state



- $\boldsymbol{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
  - $\boldsymbol{tr}(\boldsymbol{\rho}) = \frac{1}{2} < 1 \Rightarrow$ A mixed state
  - We are **uncertain** about the exact state the system is in…
  - The measurement (on system 2) outcome is b with probability $\frac{1}{2}$, corresponding to the state and the probability distribution of system 1…

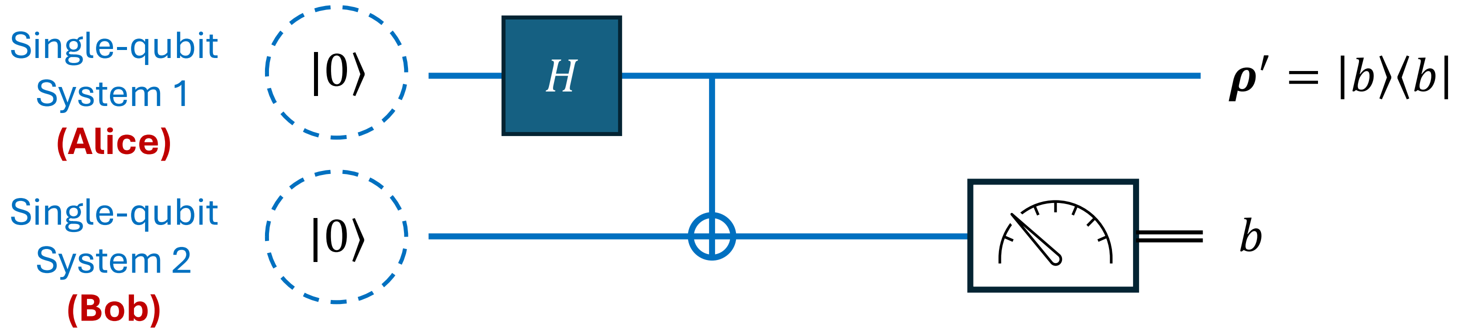
What if Bob tells the measurement outcome to Alice?

Pure state and Mixed state



- $\boldsymbol{\rho}' = |b\rangle\langle b|$

Pure state and Mixed state

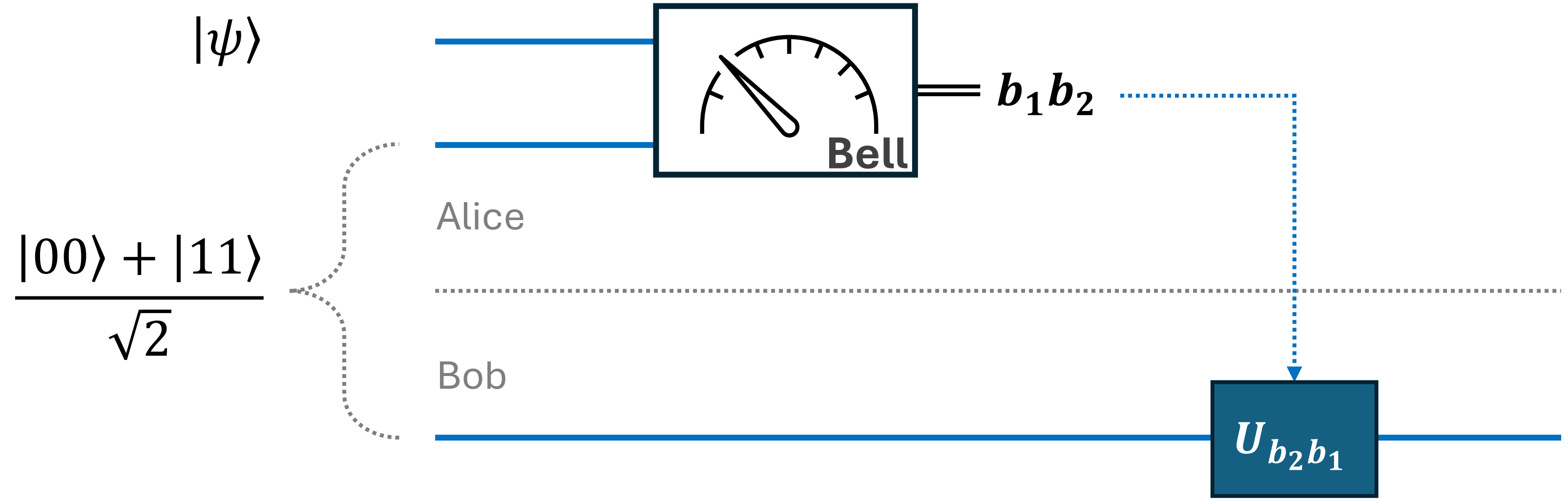


- $\boldsymbol{\rho}' = |b\rangle\langle b|$

- If Bob sends b to Alice, then $\boldsymbol{\rho}' = |b\rangle\langle b|$.
- If not, $\boldsymbol{\rho}' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ in Alice's view.
  (No information about b at all!)

Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state $|\psi\rangle$.



Total states:

$|\phi_0\rangle = \sum_{b_1,b_2\in\{0,1\}} |\beta_{b_1 b_2}\rangle \boldsymbol{U}^\dagger_{\boldsymbol{b_2 b_1}} |\psi\rangle$

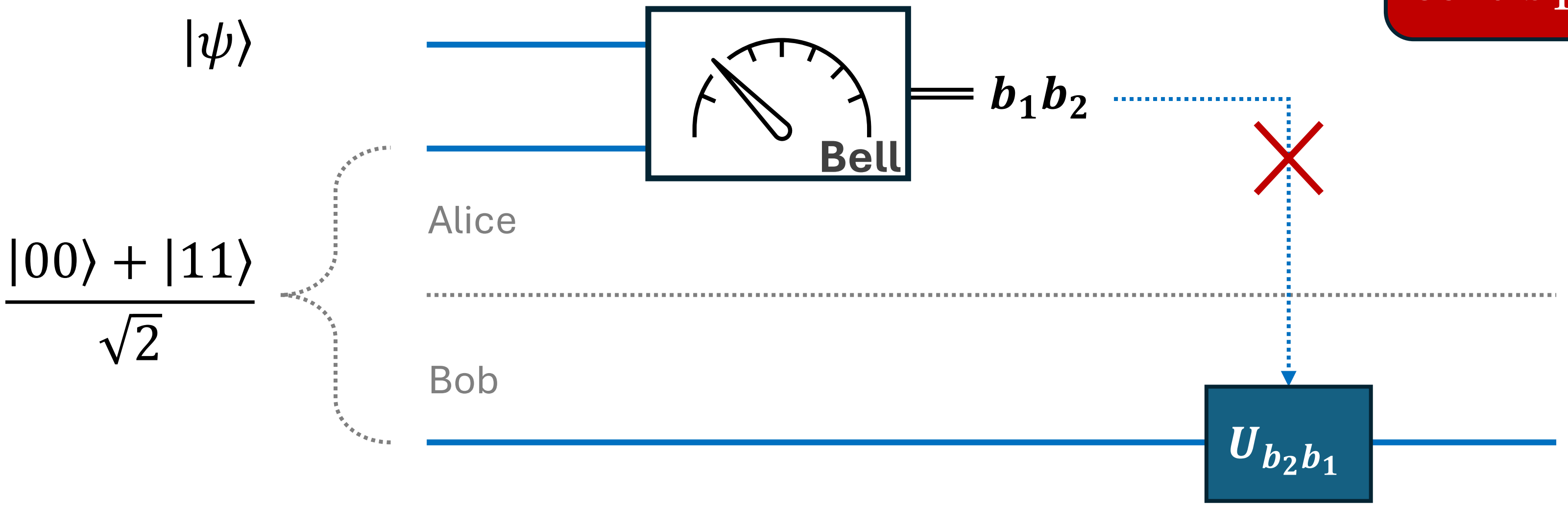
$|\phi_1\rangle = |\beta_{b_1 b_2}\rangle \boldsymbol{U}^\dagger_{\boldsymbol{b_2 b_1}} |\psi\rangle$

$|\phi_2\rangle = |\beta_{b_1 b_2}\rangle \otimes |\psi\rangle$

Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state $|\psi\rangle$.

What if Alice does not send $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$ to Bob?



Total states:

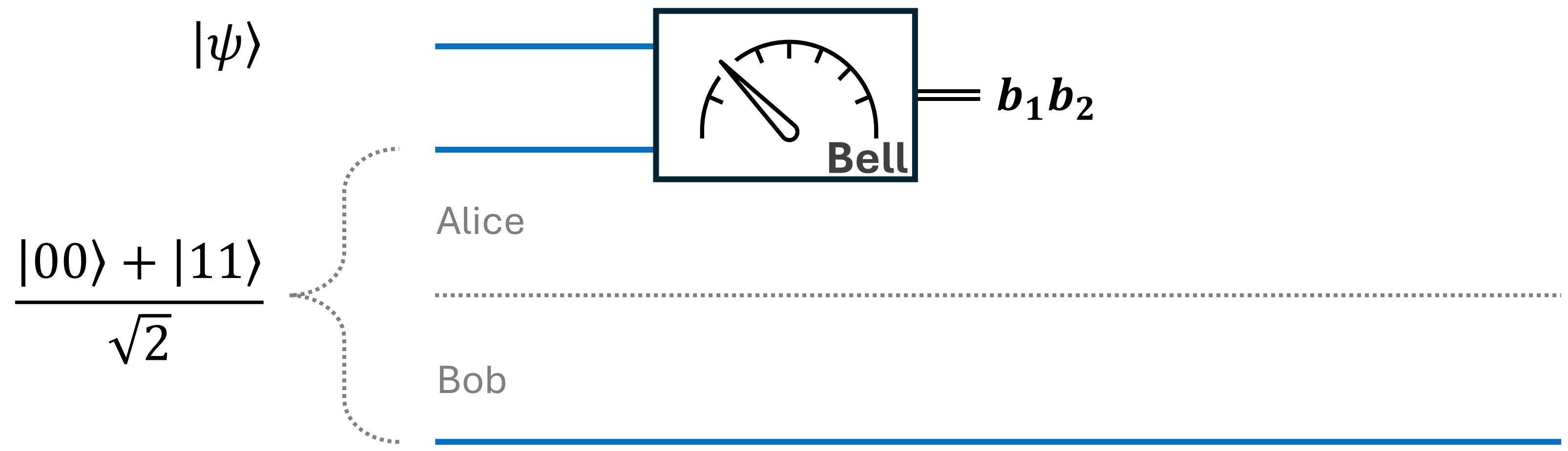
$|\phi_0\rangle = \sum_{b_1,b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle \boldsymbol{U^\dagger_{b_2 b_1}} |\psi\rangle$

$|\phi_1\rangle = |\beta_{b_1 b_2}\rangle \boldsymbol{U^\dagger_{b_2 b_1}} |\psi\rangle$

$|\phi_2\rangle = |\beta_{b_1 b_2}\rangle \otimes |\psi\rangle$

Another look at Quantum Teleportation

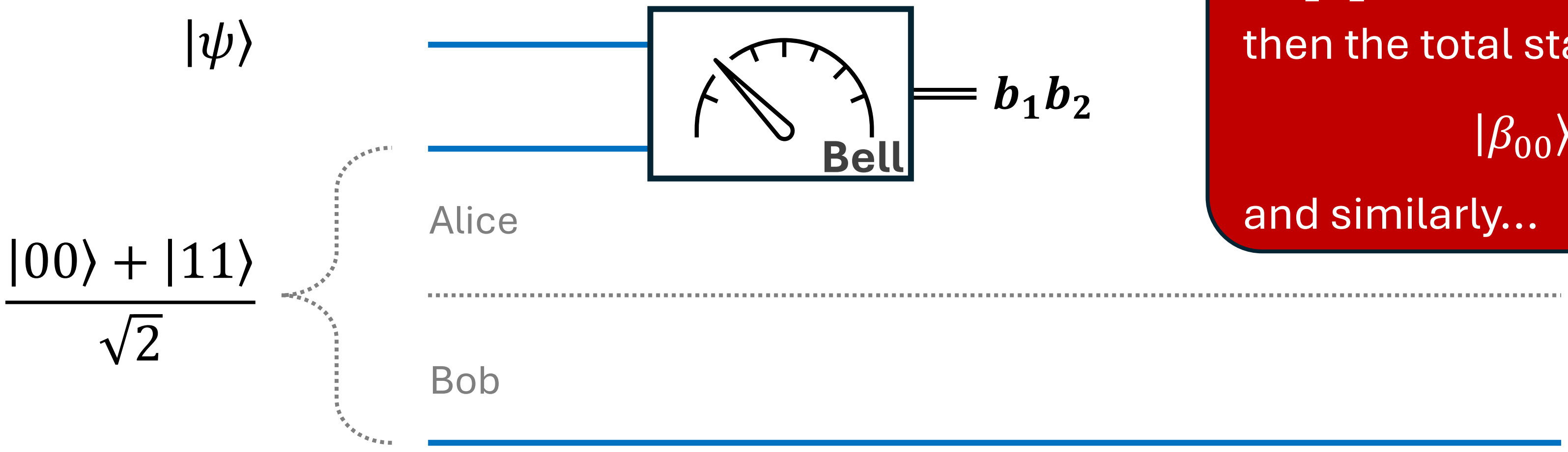
- Quantum teleportation: Transmit a single-qubit state $|\psi\rangle$.



Total states: $|\phi_0\rangle = \sum_{b_1,b_2\in\{0,1\}} |\beta_{b_1 b_2}\rangle U^\dagger_{b_2 b_1} |\psi\rangle$ $\quad |\phi_1\rangle = ?$

Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state $|\psi\rangle$.



If $b_1 b_2 = 00$ (with probability $\frac{1}{4}$), then the total state is

$$|\beta_{00}\rangle \boldsymbol{U}_{00}^{\dagger} |\psi\rangle,$$

and similarly…

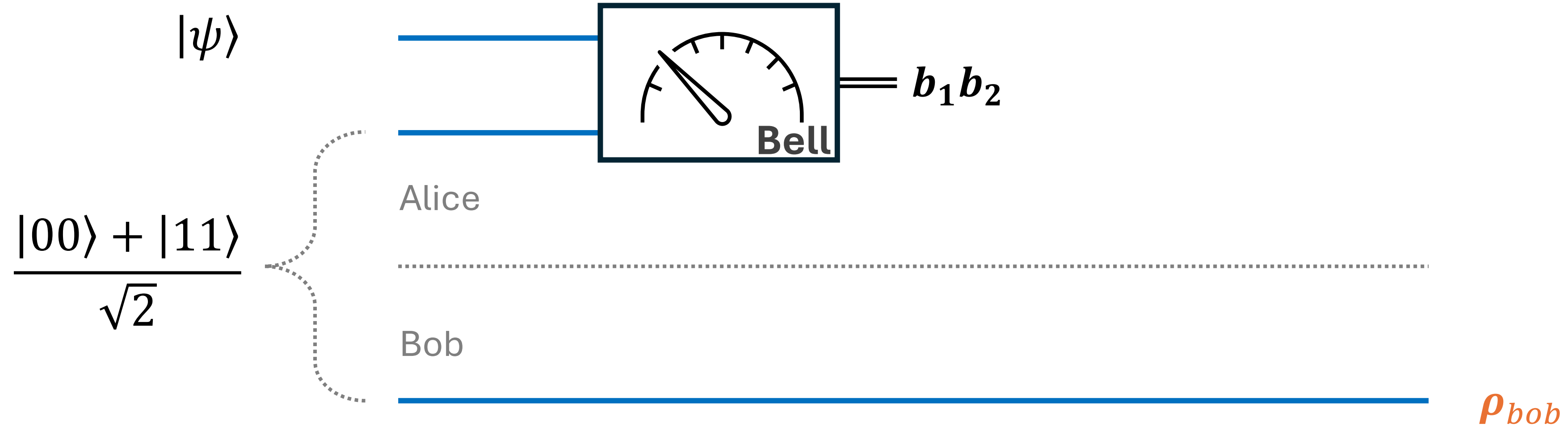
Total states: $|\phi_0\rangle = \sum_{b_1,b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle \boldsymbol{U}_{b_2 b_1}^{\dagger} |\psi\rangle$

$\boldsymbol{\rho}_1$ (the density operator on Bob's view after Alice's measurement)

$$= \frac{1}{4} \sum_{b_1,b_1 \in \{0,1\}} |\beta_{b_1 b_2}\rangle \boldsymbol{U}_{b_2 b_1}^{\dagger} |\psi\rangle\langle\beta_{b_1 b_2}| \boldsymbol{U}_{b_2 b_1} \langle\psi|$$

Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state $|\psi\rangle$.



$$\rho_{bob} = tr_{alice}(\rho_1) = tr_{alice}\left(\frac{1}{4}\sum_{b_1,b_1\in\{0,1\}} |\beta_{b_1b_2}\rangle U^{\dagger}_{b_2b_1}|\psi\rangle\langle\beta_{b_1b_2}|U_{b_2b_1}\langle\psi|\right)$$

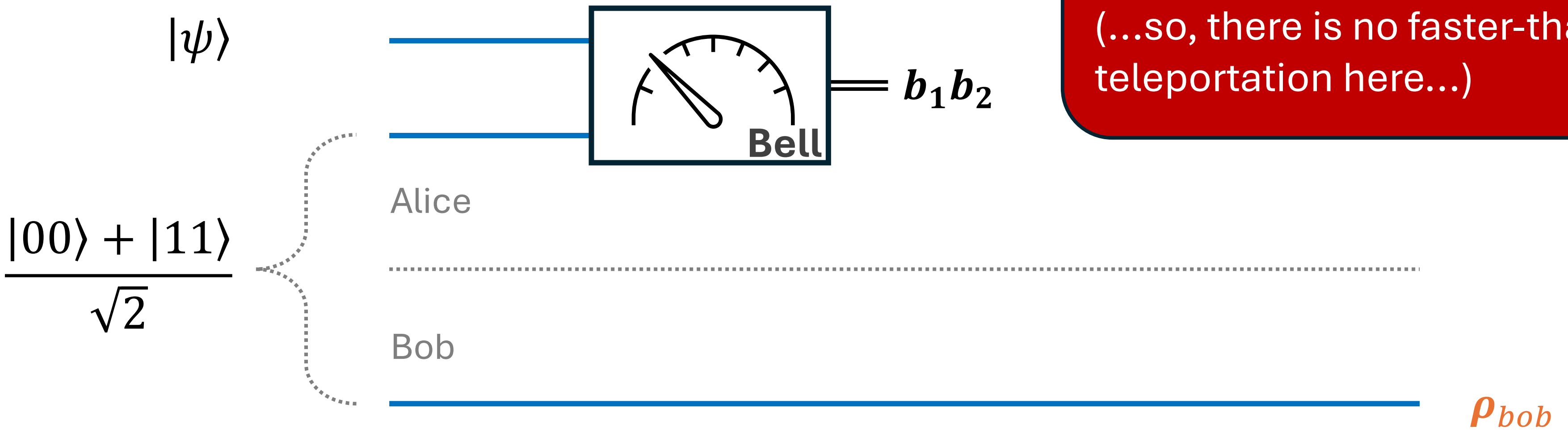
$$= I/2$$

Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state $|\psi\rangle$.

If Alice does not send $\boldsymbol{b_1 b_2}$ to Bob, then Bob learns **nothing** about $|\psi\rangle$

(…so, there is no faster-than-light teleportation here…)



$$\boldsymbol{\rho}_{bob} = \boldsymbol{tr}_{alice}(\boldsymbol{\rho}_1) = \boldsymbol{tr}_{alice}\left(\frac{1}{4}\sum_{b_1,b_1\in\{0,1\}} |\beta_{b_1 b_2}\rangle \boldsymbol{U}^{\dagger}_{b_2 b_1}|\psi\rangle\langle\beta_{b_1 b_2}|\boldsymbol{U}_{b_2 b_1}\langle\psi|\right)$$

$= \boldsymbol{I}/2$ (independent of $|\psi\rangle$)!

Reference

- **[NC00]:** Sections 2.4, 8.3.1, and 9.2.1
- **[KLM07]:** Section 3.5

- **Purification:** [NC00, Section 2.5] and [KLM07, Section 3.5.2]

Next Topics

- Quantum key distribution
- Quantum money
- Summary of this course

Trace Distance

- Let D_1 and D_2 be two probability distributions defined over the same probability space
- Define $D(m) = p_m$, where p_m is the probability that the sample is m
- Trace distance between D_1 and D_2:

$$\mathbf{TD}(D_1, D_2) := \frac{1}{2}\sum_m |D_1(m) - D_2(m)|$$

- Trace distance **measures how close the two distributions are.**
- Example: Biased coin vs Fair coin

Trace Distance

- Let ρ_1 and ρ_2 be two density operators
- Trace distance between ρ_1 and ρ_2:

$$\mathbf{TD}(\rho_1, \rho_2) \coloneqq \frac{1}{2} \boldsymbol{tr}(|\rho_1 - \rho_2|)$$

- $\mathbf{TD}(\rho_1, \rho_2)$ bound the trace distance between the measurement distributions of ρ_1 and ρ_2.
  - Namely, let D_1 and D_2 be the measurement distributions of ρ_1 and ρ_2, respectively.
  - Then $\mathbf{TD}(D_1, D_2) \leq \mathbf{TD}(\rho_1, \rho_2)$ (regardless of what measurement basis we choose...)