Quantum Computing

- Lectures 4-5 (May 7 and 8, 2025)

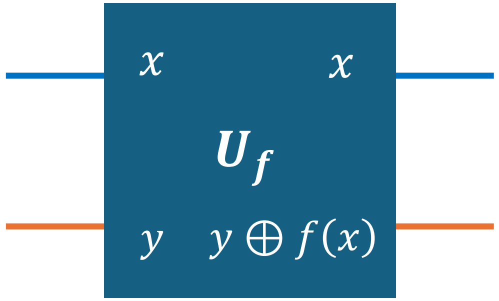
- Today:
    - Unitary operations on multi-qubit systems
    - Some examples (do it on the board)

Unitary Operations

- A quantum gate is a unitary operator ($\Leftrightarrow$ A unitary represents some quantum gate)
  - A unitary operator has **linearity:** $\boldsymbol{U}(c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2}) = c_1\boldsymbol{U}\boldsymbol{v_1} + c_2\boldsymbol{U}\boldsymbol{v_2}$

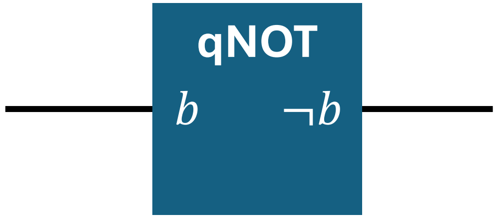
- Quantum gates operate on superposition: **Linearity**
  - View any quantum gate as a unitary linear operator (matrix)
  - Quantum gates act on superpositions according to linearity

- Make a classical computable function unitary $f \rightarrow \boldsymbol{U_f}$
  - Use **input qubits** and **ancilla qubits** to make it invertible
  - Any classical algorithm can be simulated by quantum computers



Unitary Operations on Single-Qubit States

- Single-qubit unitary:
  - Examples: qNOT, Hadamard transform, …



$$\mathbf{qNOT}(|b\rangle) \to |\neg b\rangle$$

$$\boldsymbol{H}(|b\rangle) \to \frac{|0\rangle + (-1)^b|1\rangle}{\sqrt{2}}$$

Unitary Operations on Single-Qubit States

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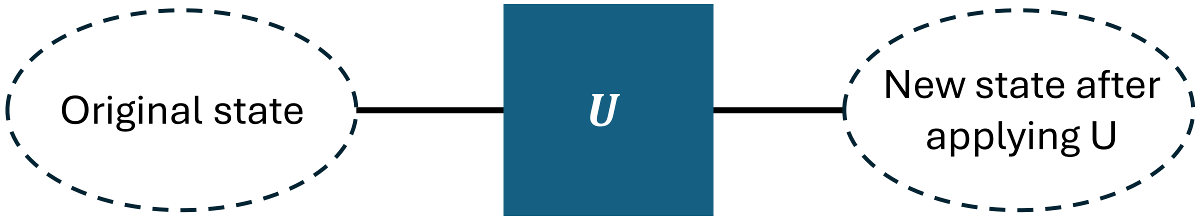
Some **isolated** single-qubit system



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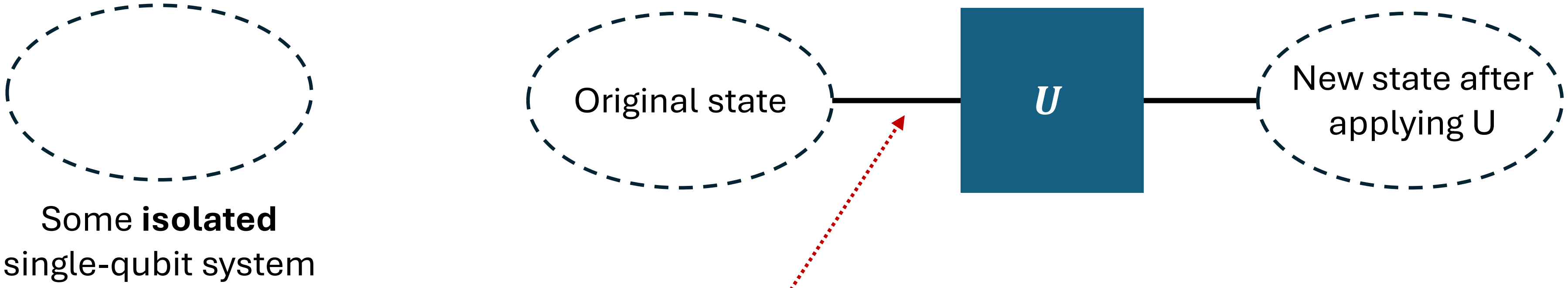

Misunderstand: The "wire" here **does not** represent a real wire!

Instead, it just visually **tracks the state of the system** through time.

Unitary Operations on Single-Qubit States

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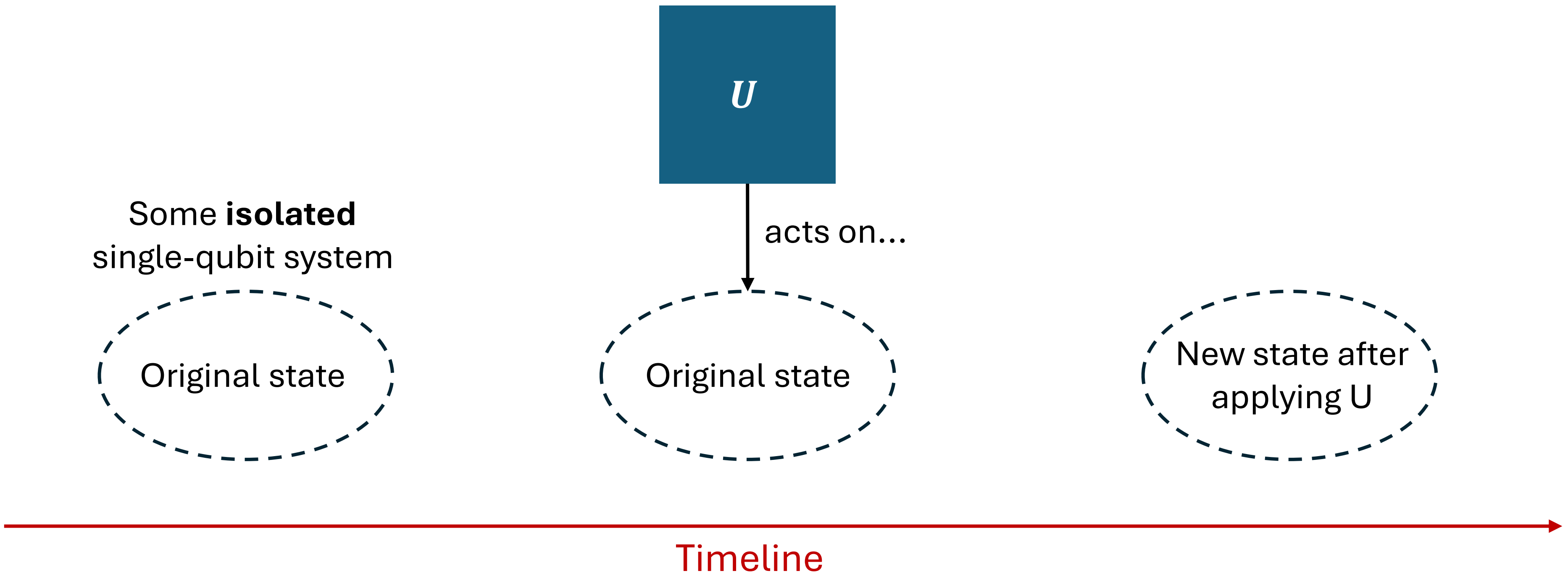
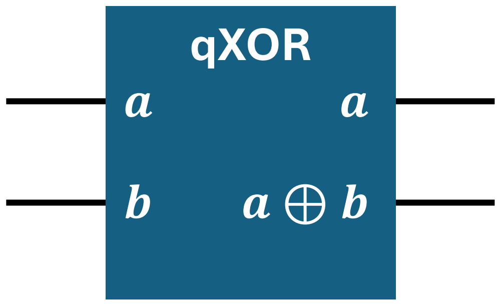
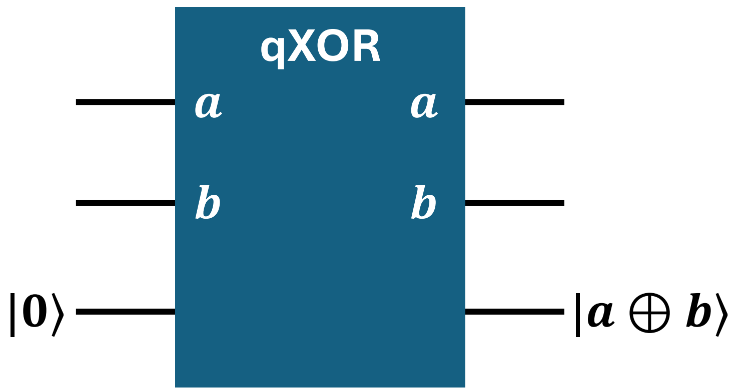
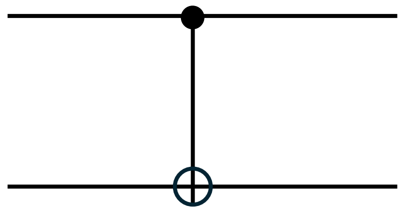


Unitary Operations on Multi-Qubit States

- Multi-qubit unitary:
  - Examples: qXOR, CNOT, …







CNOT

Unitary Operations on Multi-Qubit States

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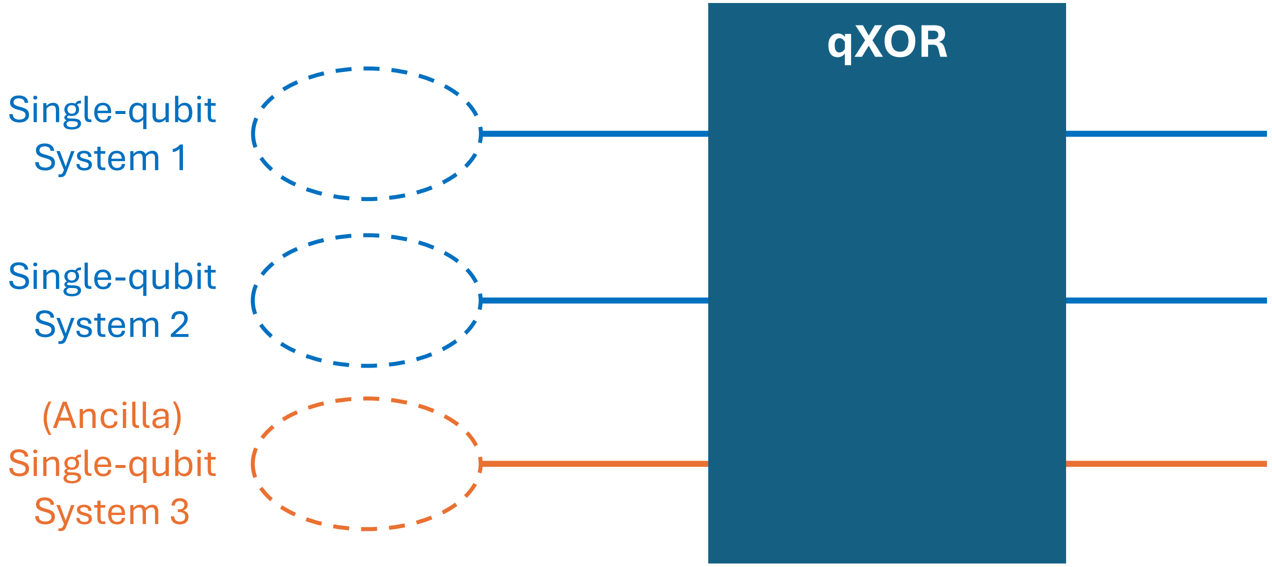

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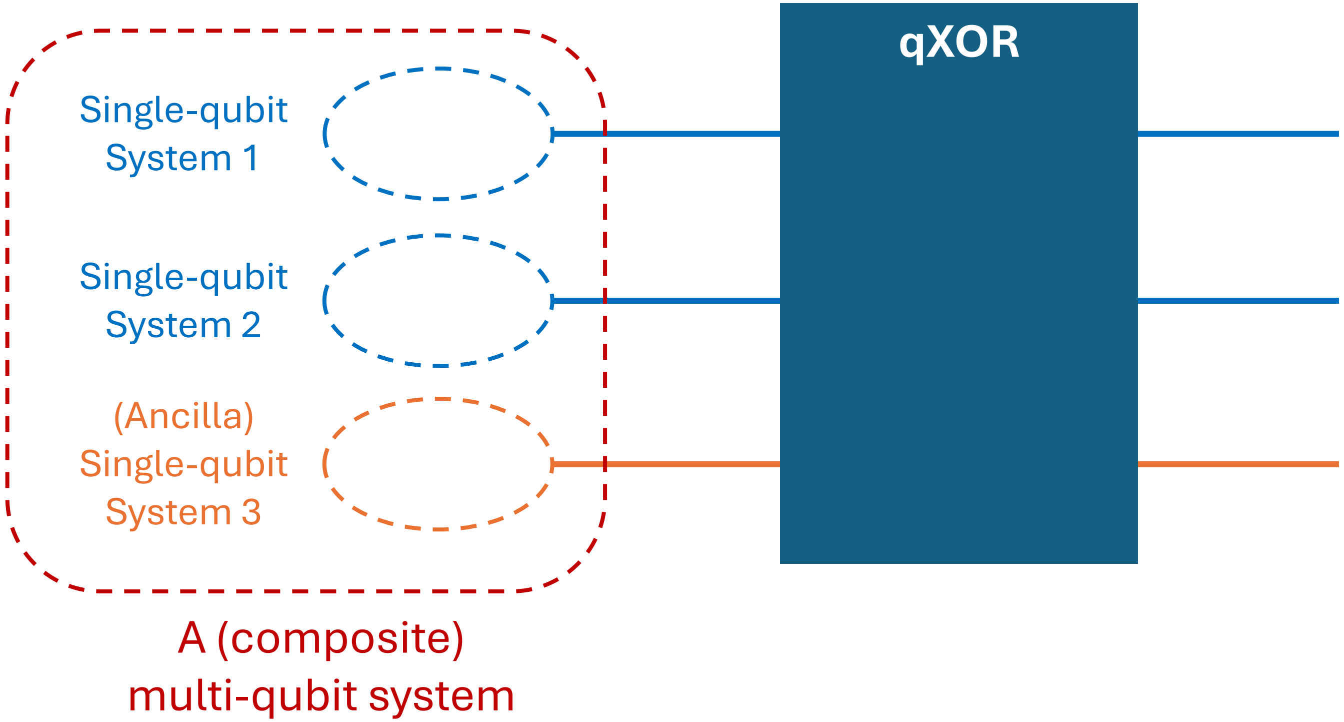

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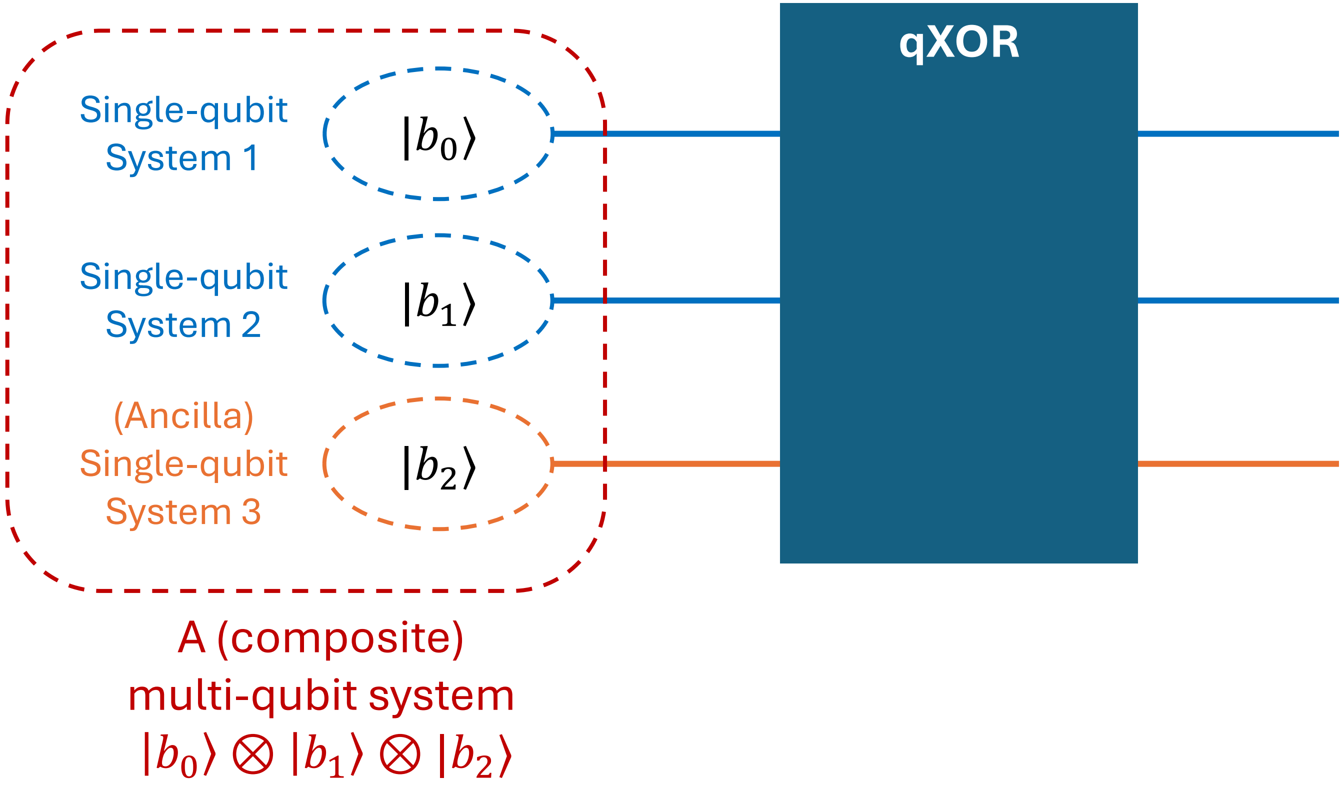
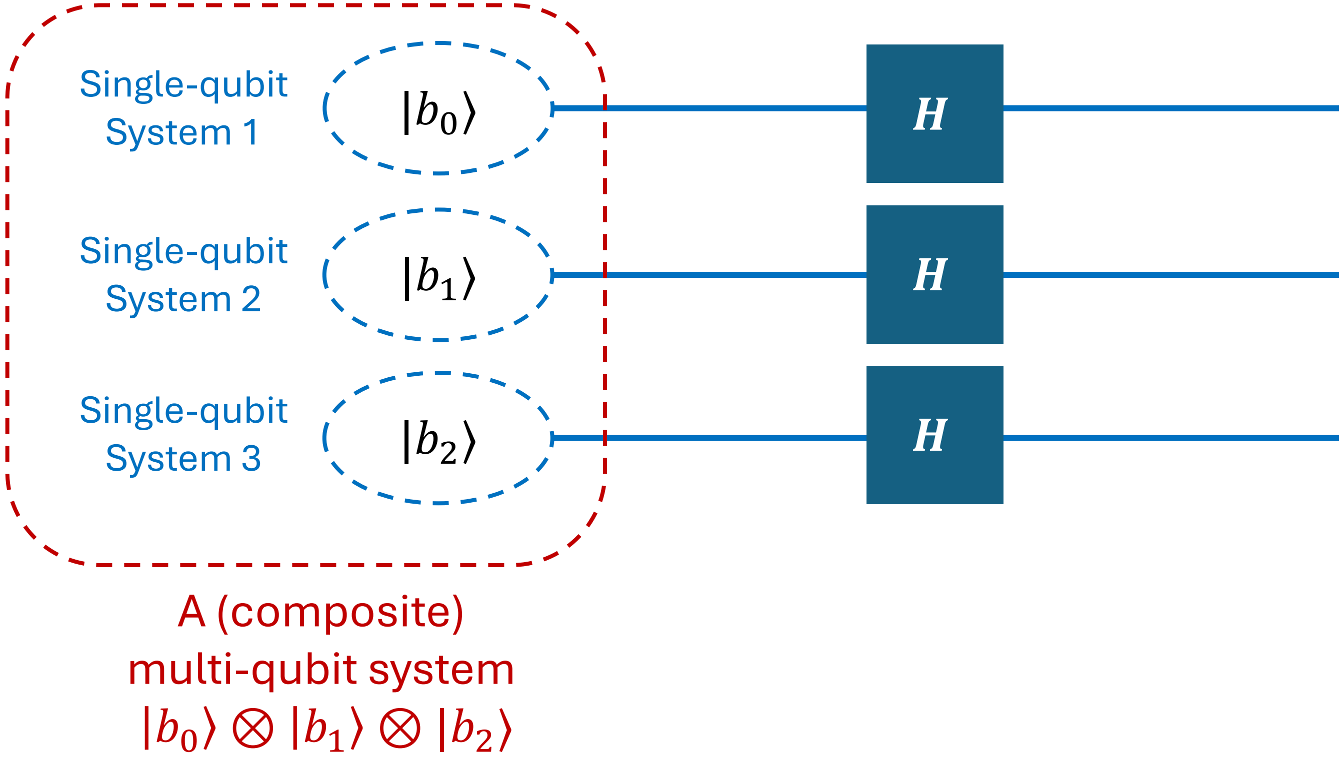


Unitary Operations on Multi-Qubit States

- Multi-qubit unitary:
  - Examples: qXOR, CNOT, **Parallel action of Hadamard gates**...



Unitary Operations on Multi-Qubit States

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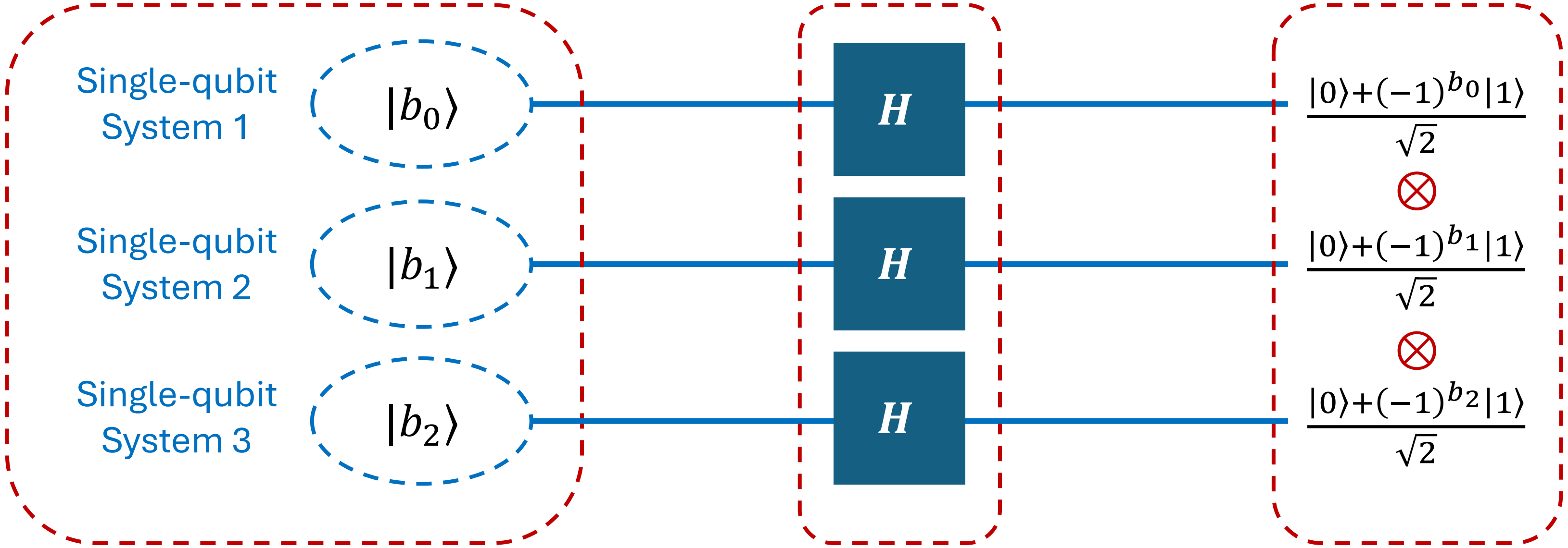


A (composite) multi-qubit system $|b_0\rangle \otimes |b_1\rangle \otimes |b_2\rangle$

$\boldsymbol{H}^{\otimes 3}$ (Parallel H acts on 3-qubit system)

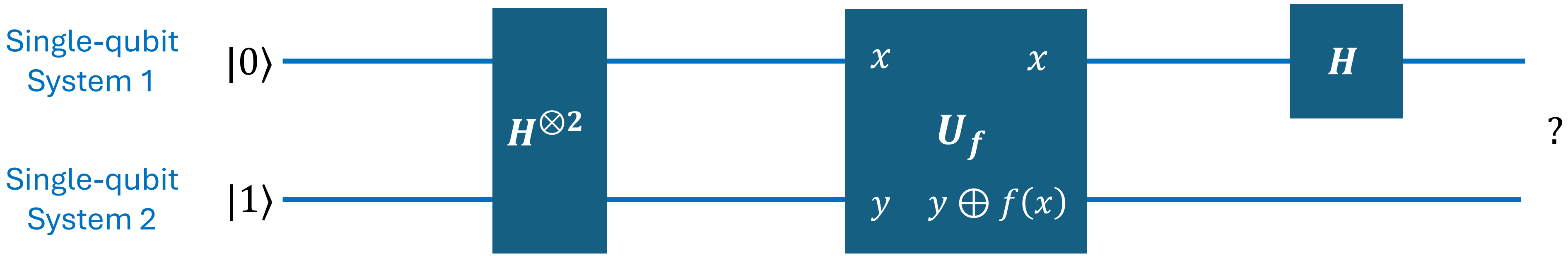
The state of the composite system after applying $\boldsymbol{H}^{\otimes 3}$

Unitary Operations on Multi-Qubit States

- Multi-qubit unitary:
  - **Parallel action of Hadamard gates…**

Deutsch's Algorithm

- Let f be a bit function...
- (Do it on the board)



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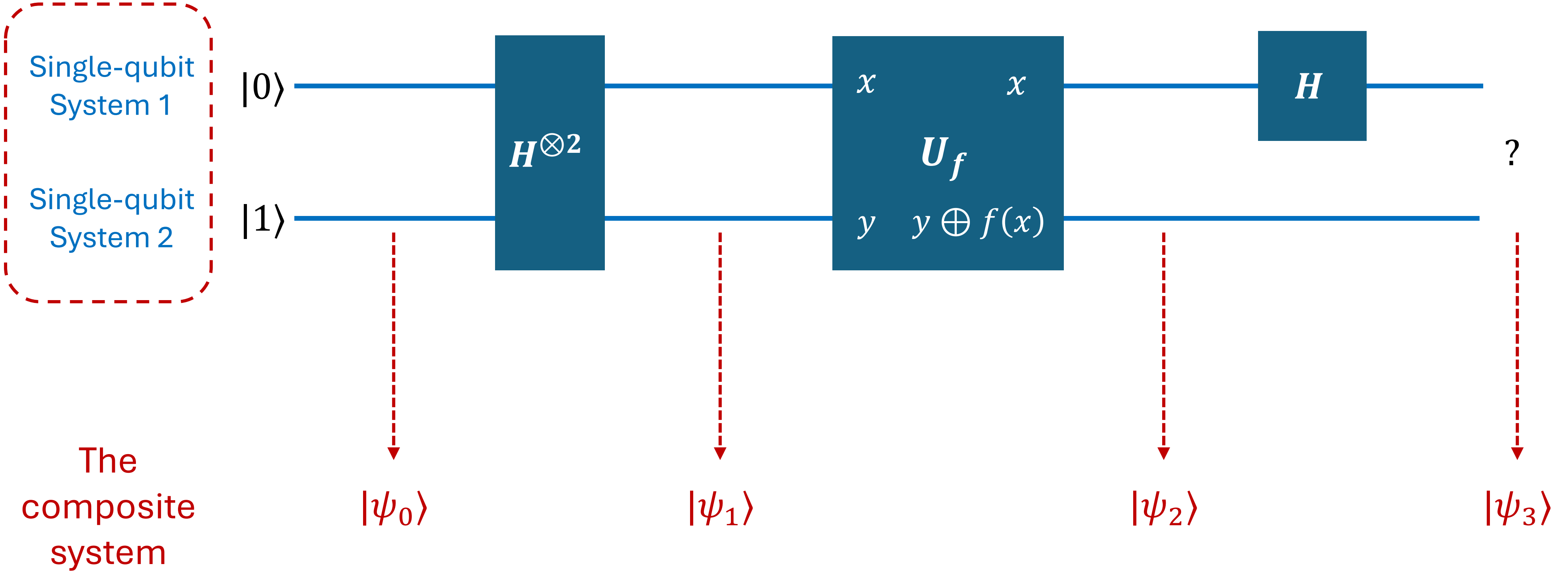

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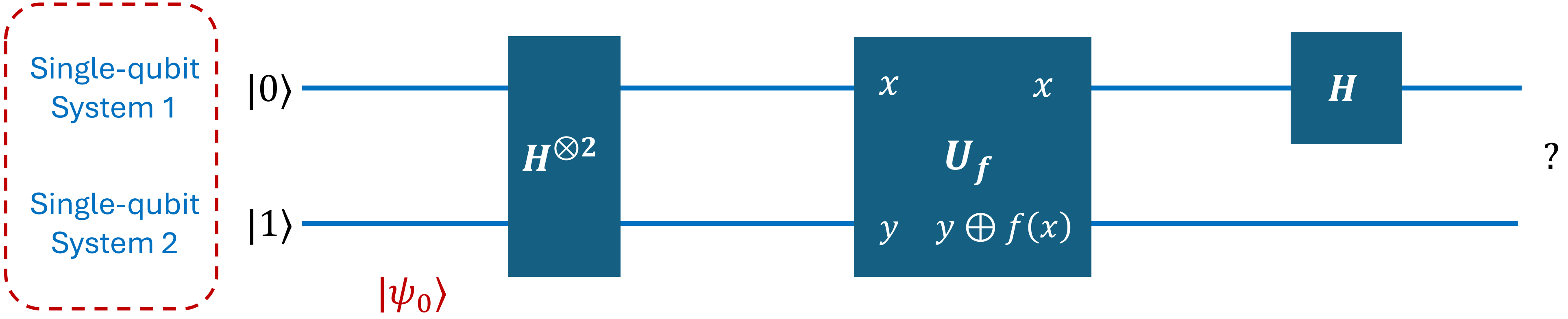

The composite system

$$|\psi_0\rangle = |01\rangle = |0\rangle \otimes |1\rangle$$

Deutsch's Algorithm

- Let f be a bit function...
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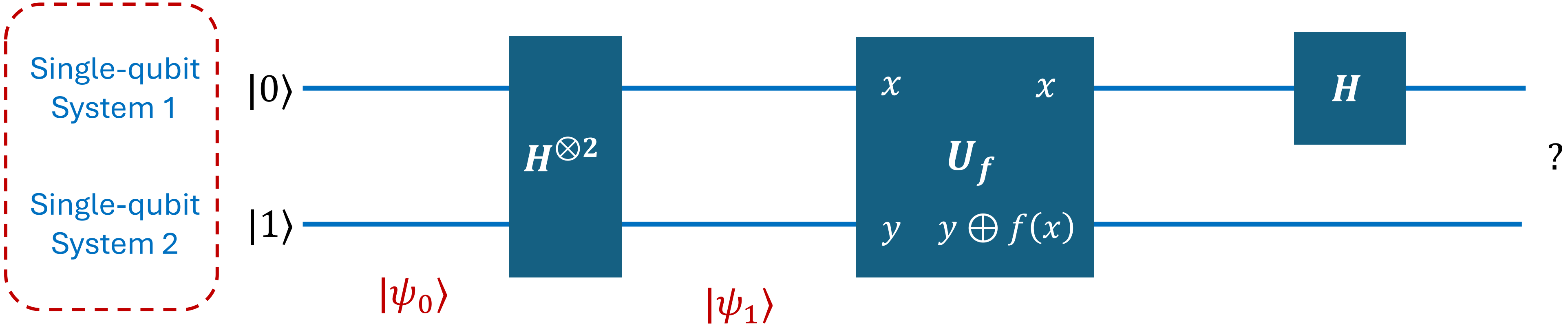

The composite system

$$|\psi_1\rangle = \boldsymbol{H}^{\otimes 2}|\psi_0\rangle = \left(\frac{|0\rangle + |1\rangle}{2}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{2}\right)$$

Deutsch's Algorithm

- Let f be a bit function…
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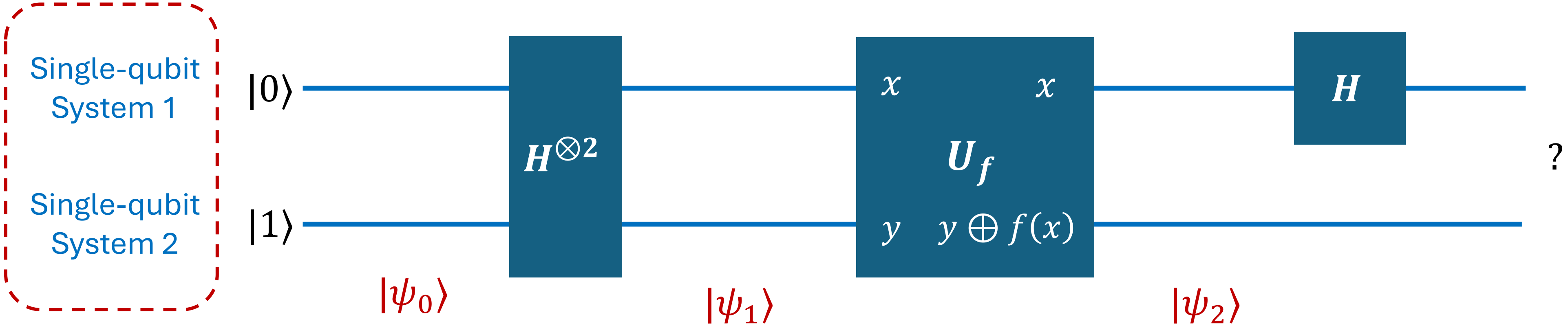
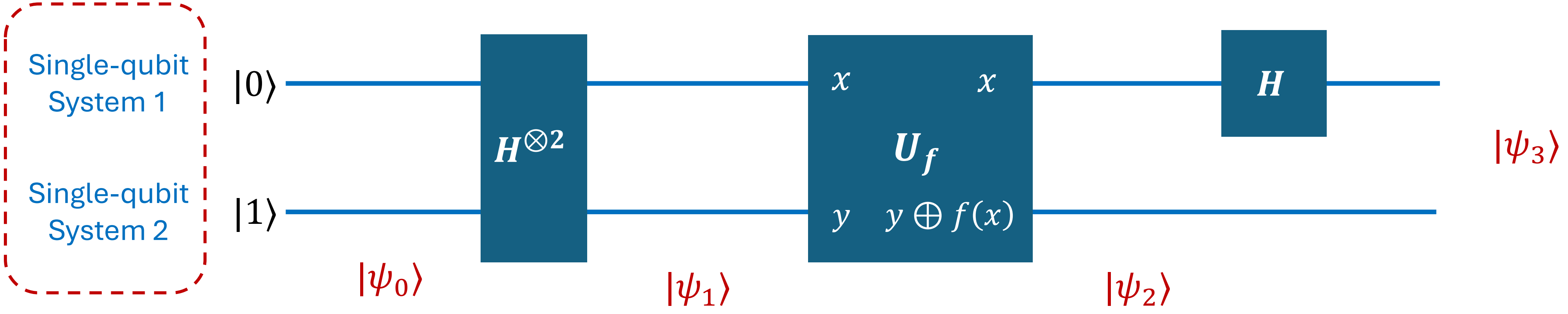


The composite system

$$|\psi_2\rangle = \boldsymbol{U}_f|\psi_1\rangle = \left(\frac{|0\rangle + (-1)^{f(0)\oplus f(1)}|1\rangle}{2}\right) \otimes \left((-1)^{f(0)}\left(\frac{|0\rangle - |1\rangle}{2}\right)\right)$$

Deutsch's Algorithm

- Let f be a bit function...
- (Do it on the board)



The composite system

$$|\psi_3\rangle = (\boldsymbol{H} \otimes \boldsymbol{I})|\psi_2\rangle = (|f(0) \oplus f(1)\rangle) \otimes \left((-1)^{f(0)}\left(\frac{|0\rangle - |1\rangle}{2}\right)\right)$$

Deutsch's Algorithm

- Let f be a bit function...
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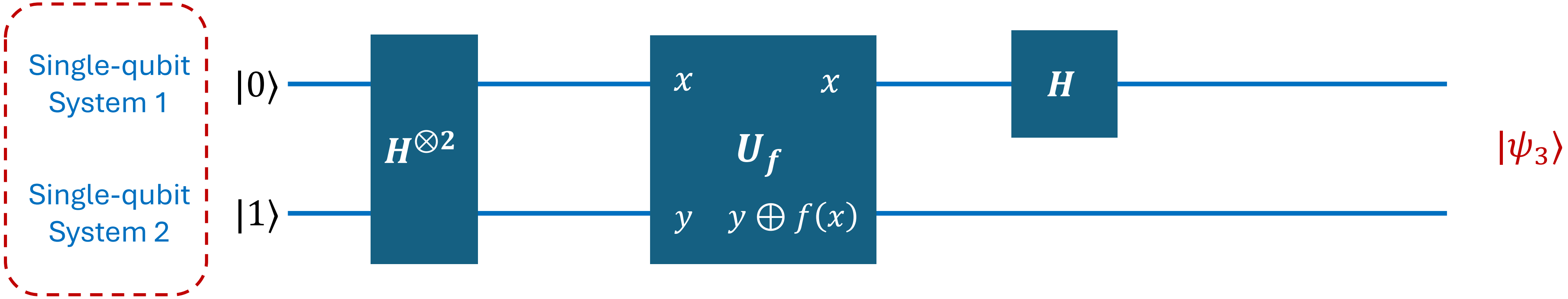


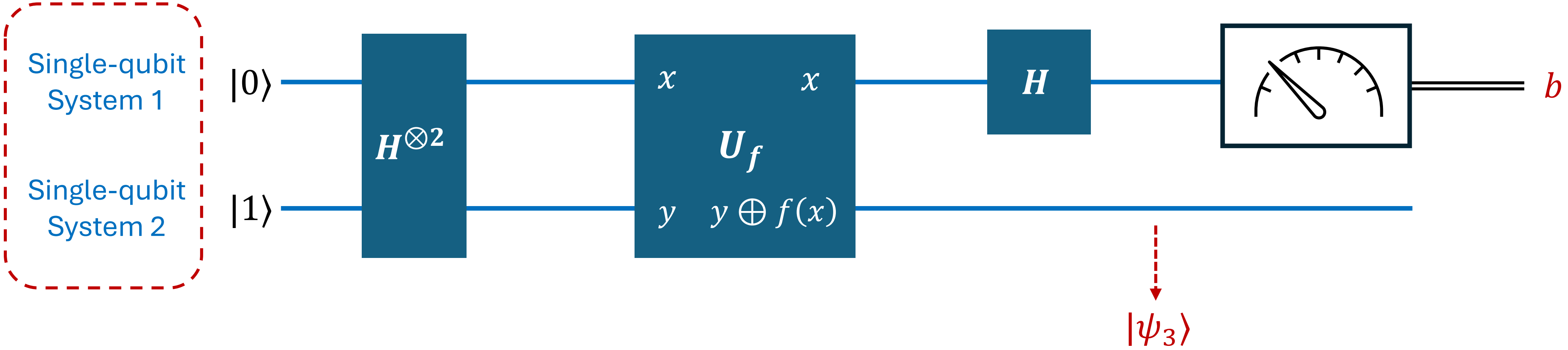
The composite system

$$|\psi_3\rangle = \pm|f(0) \oplus f(1)\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

(We just let $(-1)^{f(0)} = \pm$, which does not change the measurement outcome)

Deutsch's Algorithm

- Let f be a bit function...
- (Do it on the board)

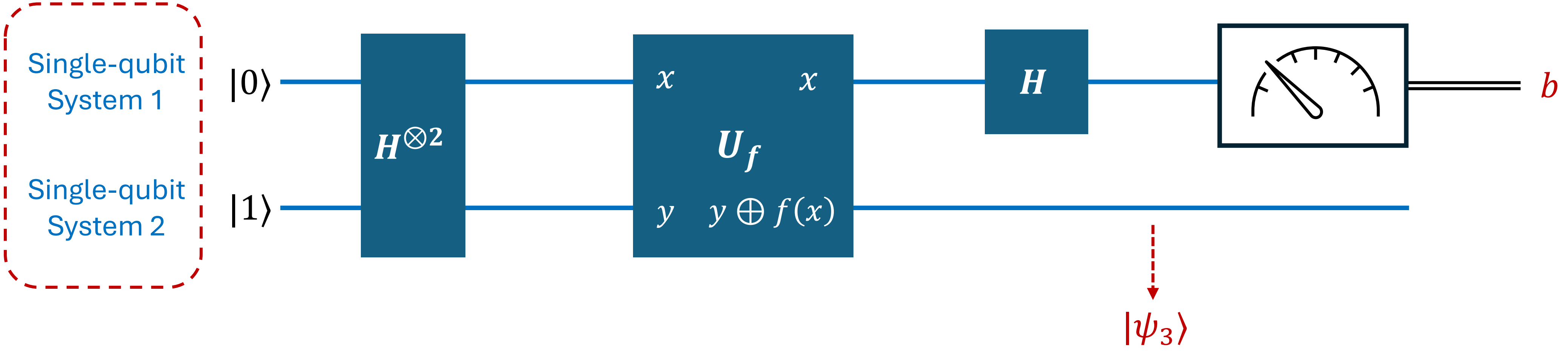


The composite system

$$|\psi_3\rangle = \pm|f(0) \oplus f(1)\rangle \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right] \xRightarrow{\text{Measure the first system}} b = f(0) \oplus f(1)$$

Deutsch's Algorithm

- Let f be a bit function…
- (Do it on the board)



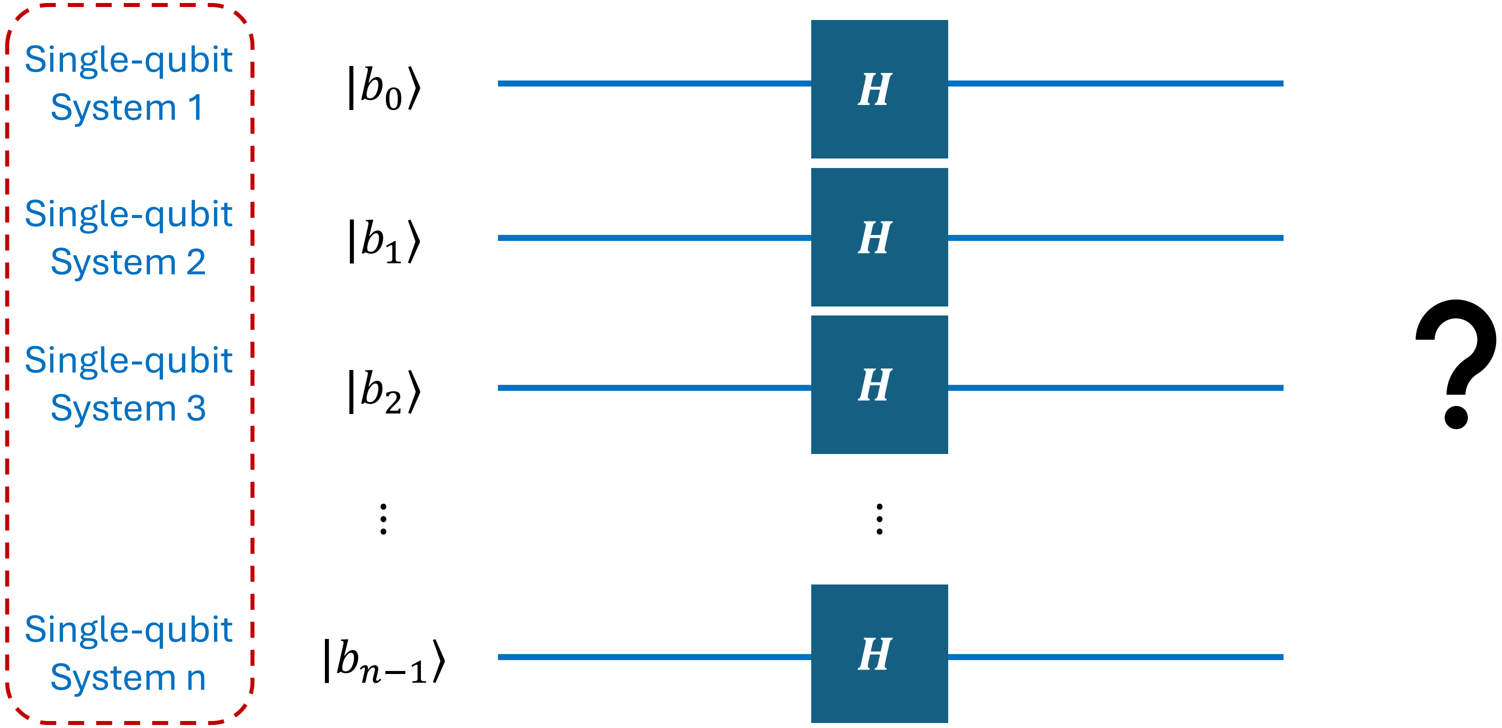
The composite system

Measure the first system

$$|\psi_3\rangle \Longrightarrow b = f(0) \oplus f(1)$$

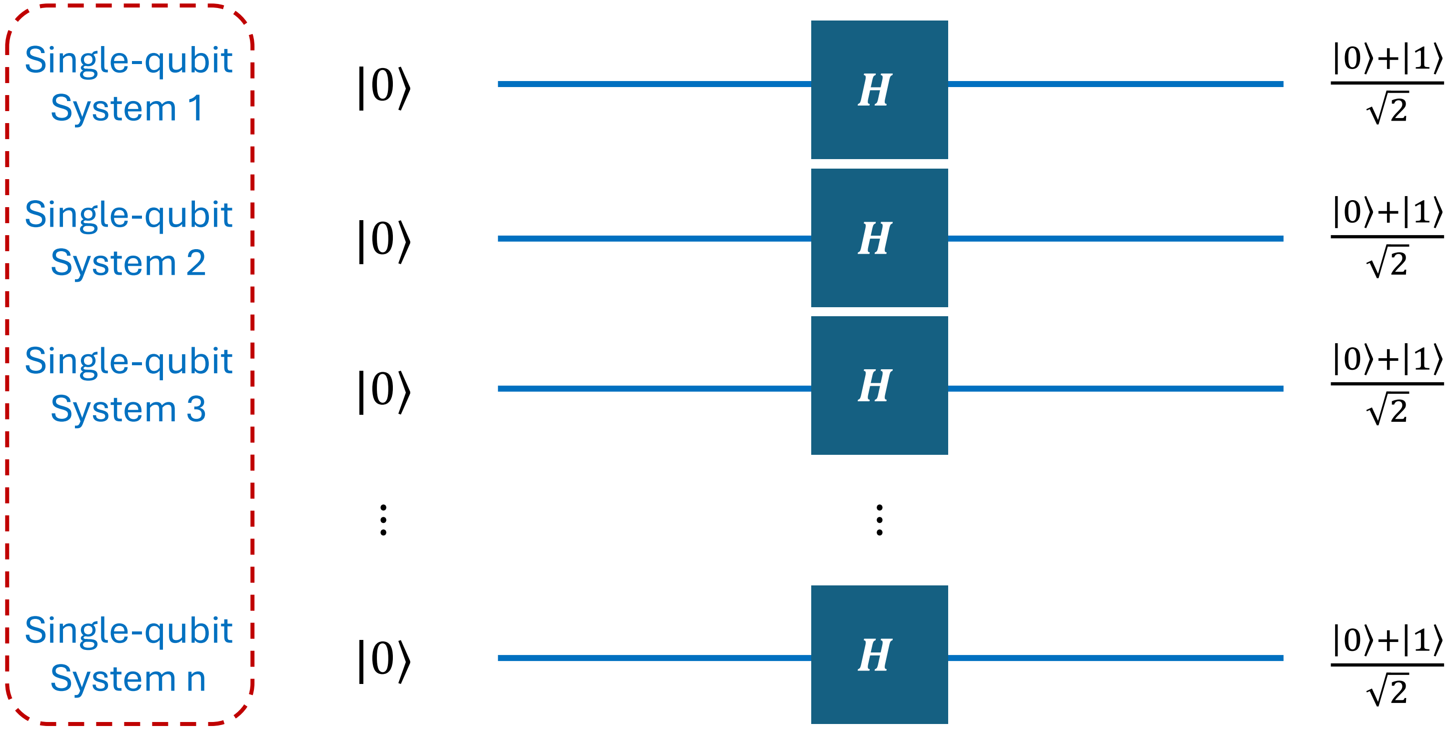
One (quantum) evaluation on f and get $f(0) \oplus f(1)$

Parallel Hadamard Gates



$|b_0\rangle \otimes |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_{n-1}\rangle$

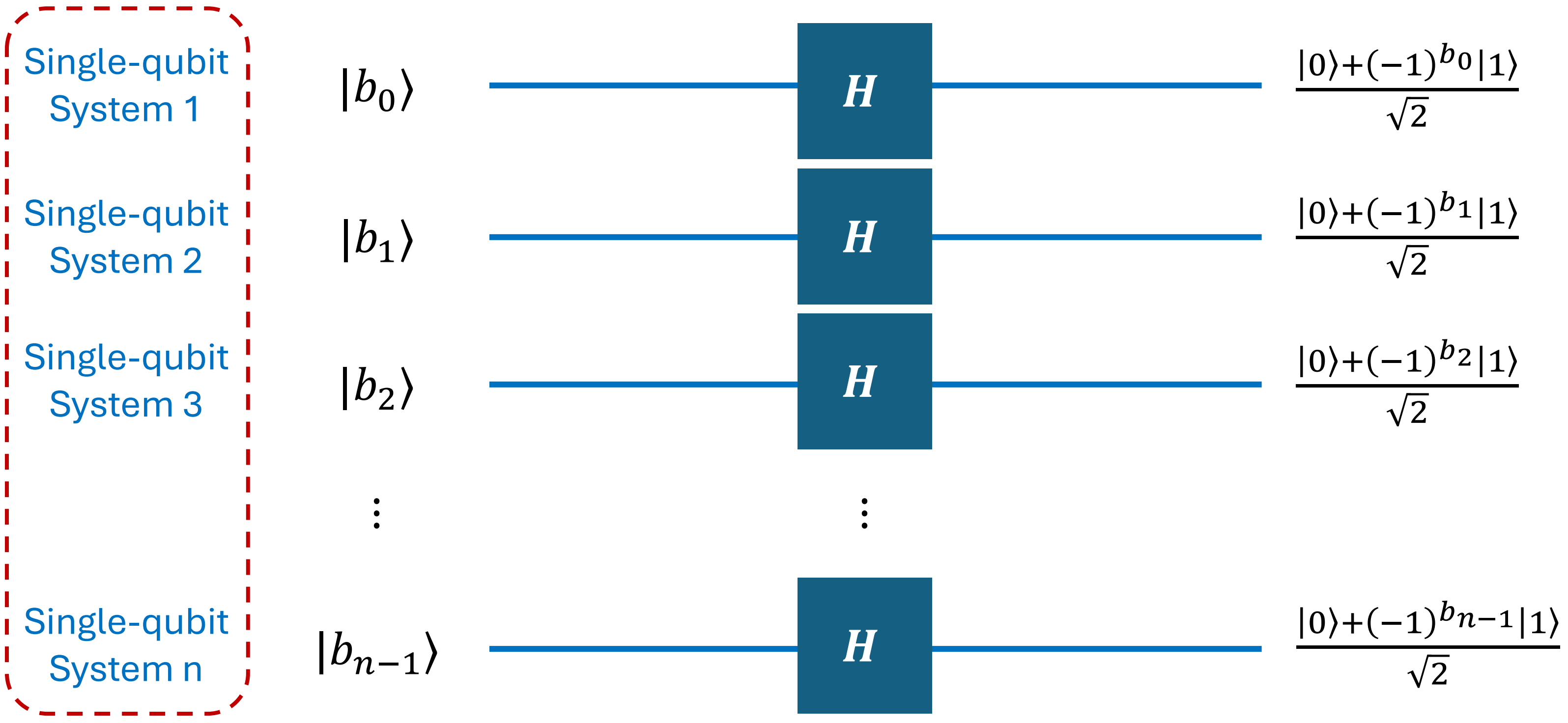
Parallel Hadamard Gates



$$= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |\boldsymbol{x}\rangle$$

$$|b_0\rangle \otimes |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_{n-1}\rangle$$

Parallel Hadamard Gates



$|b_0\rangle \otimes |b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_{n-1}\rangle$

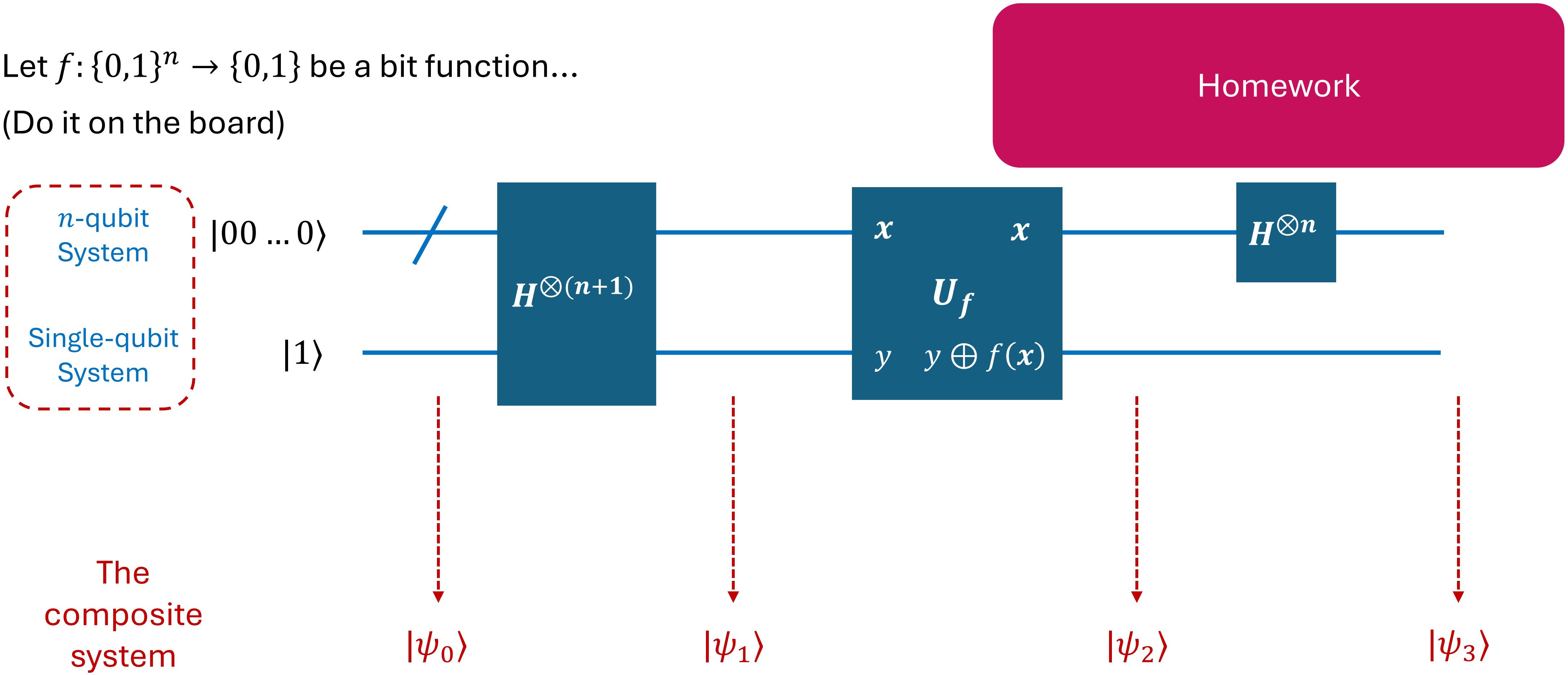
Let $\boldsymbol{b} := b_{n-1}b_{n-2}\ldots b_0$ be the classical bit string

$$\boldsymbol{H}^{\otimes n}|\boldsymbol{b}\rangle$$

$$= \sum_{x\in\{0,1\}^n} \frac{(-1)^{\boldsymbol{x}^T\boldsymbol{b}}}{\sqrt{2^n}}|\boldsymbol{x}\rangle$$

The Deutsch-Jozsa Algorithm

- Let $f: \{0,1\}^n \to \{0,1\}$ be a bit function…
- (Do it on the board)



References

- **[NC00]** *Quantum Computation and Quantum Information.*
  - Sections 1.4.3
- **[KLM07]** *An Introduction to Quantum Computing.*
  - Sections 6.2, 6.3, and 6.4
- **[RP11]** *Quantum Computing: A Gentle Introduction.*
  - Section 7.3.1

Next Topic

- **No lectures in the next week!**

- **First homework set** (will be announced in the Moodle system soon)**:**
  - Submit your **handwritten solutions (photos, scanned pdf, etc.), or typeset in LaTeX**
  - Solutions may be found in the textbook…
  - But please **include all intermediate equations and their explanation**