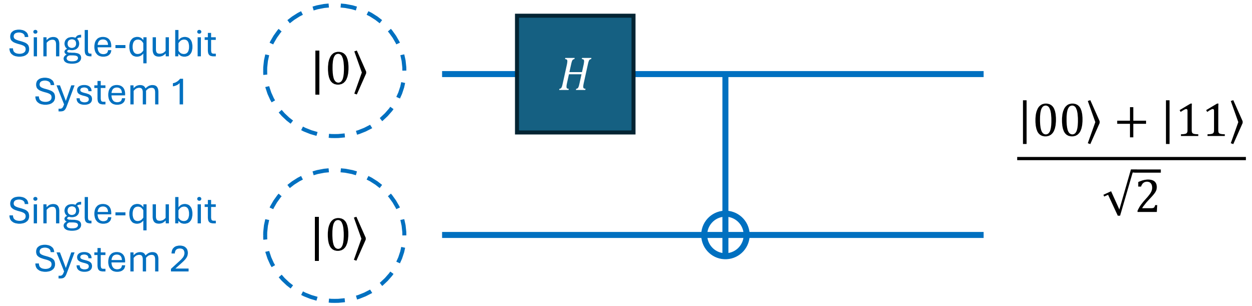
Quantum Computing

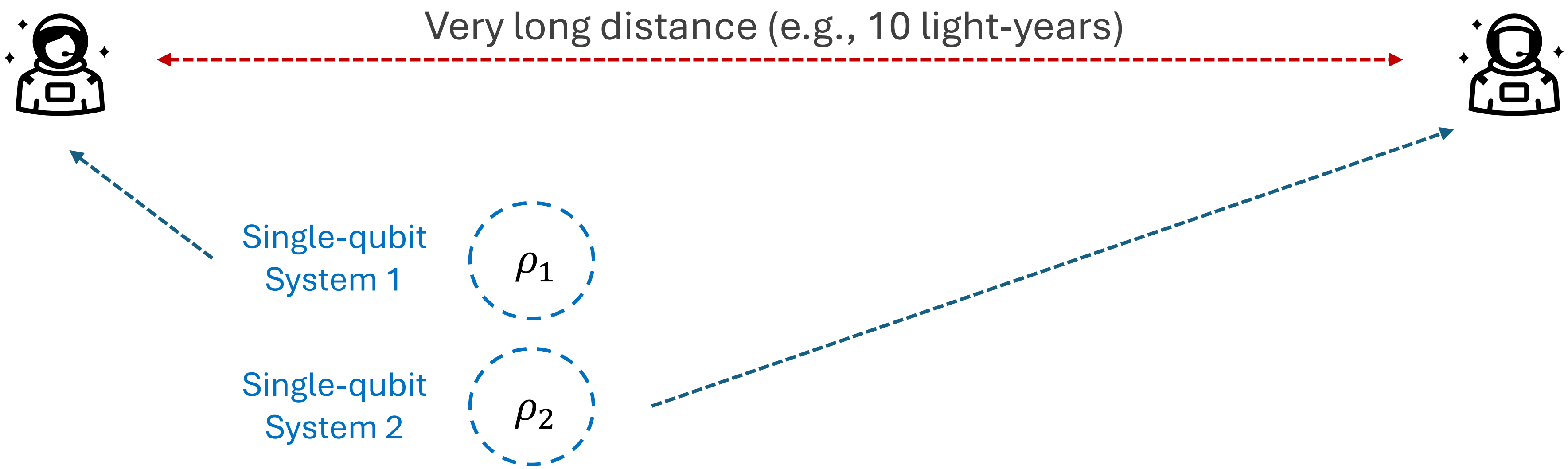
- Lectures 9 and 10 (June 4-5, 2025)

- Today:
  - Superdense coding
  - Quantum teleportation

Action at a Distance



Action at a Distance



Total state: $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Action at a Distance

Very long distance (e.g., 10 light-years)

Single-qubit System 1

ρ_1

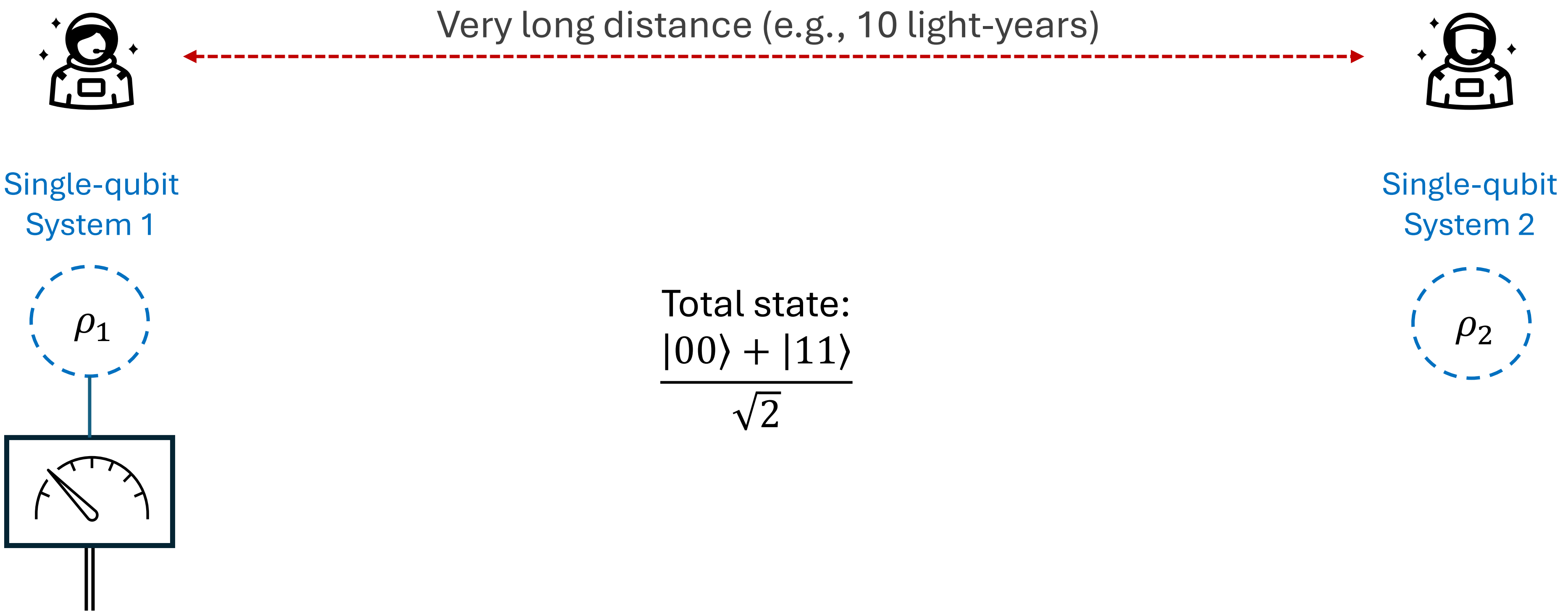
Single-qubit System 2

ρ_2

Total state:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

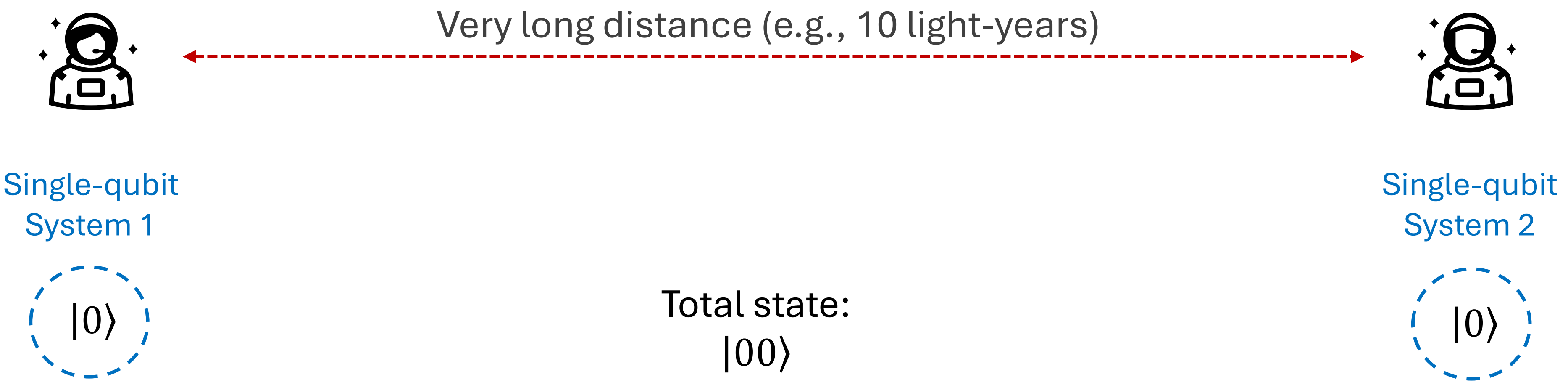
Action at a Distance



Total state:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Action at a Distance



- Application: Superdense coding

Pauli Matrices

- Pauli matrices:

$$\boldsymbol{X} = \sigma_1 \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
(Pauli-**X**)

$$\boldsymbol{Y} = \sigma_2 \coloneqq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$
(Pauli-**Y**)

$$\boldsymbol{Z} = \sigma_3 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
(Pauli-**Z**)

- Some facts:
  - Pauli-X is the qNOT gate (in the computational basis)
  - $\sigma_j^2 = I$ for $j = 1,2,3$

Pauli Matrices

- (Extended) Pauli matrices:

$$\boldsymbol{I} = \sigma_0 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{X} = \sigma_1 \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{Y} = \sigma_2 \coloneqq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \boldsymbol{Z} = \sigma_3 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(Pauli-**X**) (Pauli-**Y**) (Pauli-**Z**)

- Some facts:
  - Pauli-X is the qNOT gate (in the computational basis)
  - $\sigma_j^2 = I$ for $j = 0,1,2,3$

Superdense Coding

- Consider the four matrices

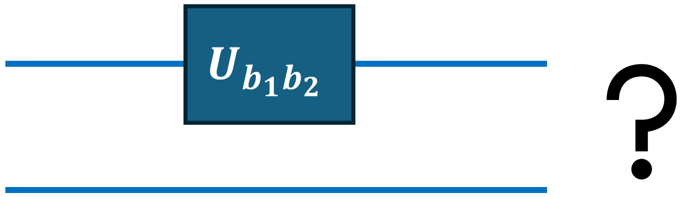
$$I = \sigma_0 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \sigma_1 \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \sigma_3 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let's define:

$$U_{b_1 b_2} \coloneqq Z^{b_2} X^{b_1} \begin{cases} U_{00} = I \\ U_{01} = X \\ U_{10} = Z \\ U_{11} = Z \cdot X \end{cases}$$

- Small Exercise:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



Superdense Coding

- Consider the four matrices

$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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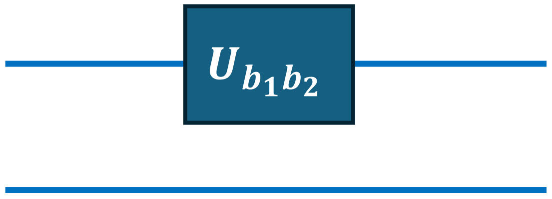


$$|\beta_{b_1 b_2}\rangle$$

Superdense Coding

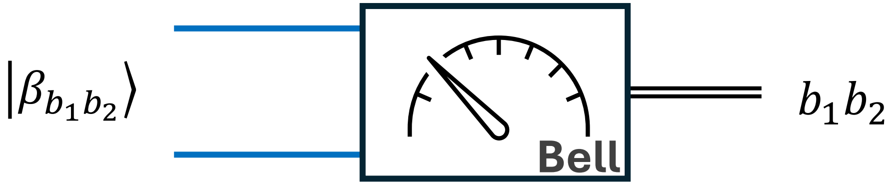
- Consider the four matrices

$$\boldsymbol{I} = \sigma_0 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{X} = \sigma_1 \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{Z} = \sigma_3 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \boldsymbol{Z} \cdot \boldsymbol{X} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let's define:

$$\boldsymbol{U}_{\boldsymbol{b_1 b_2}} \coloneqq \boldsymbol{Z}^{\boldsymbol{b_2}} \boldsymbol{X}^{\boldsymbol{b_1}} \begin{cases} \boldsymbol{U}_{\boldsymbol{00}} = \boldsymbol{I} \\ \boldsymbol{U}_{\boldsymbol{01}} = \boldsymbol{X} \\ \boldsymbol{U}_{\boldsymbol{10}} = \boldsymbol{Z} \\ \boldsymbol{U}_{\boldsymbol{11}} = \boldsymbol{Z} \cdot \boldsymbol{X} \end{cases}$$

- Small Exercise:

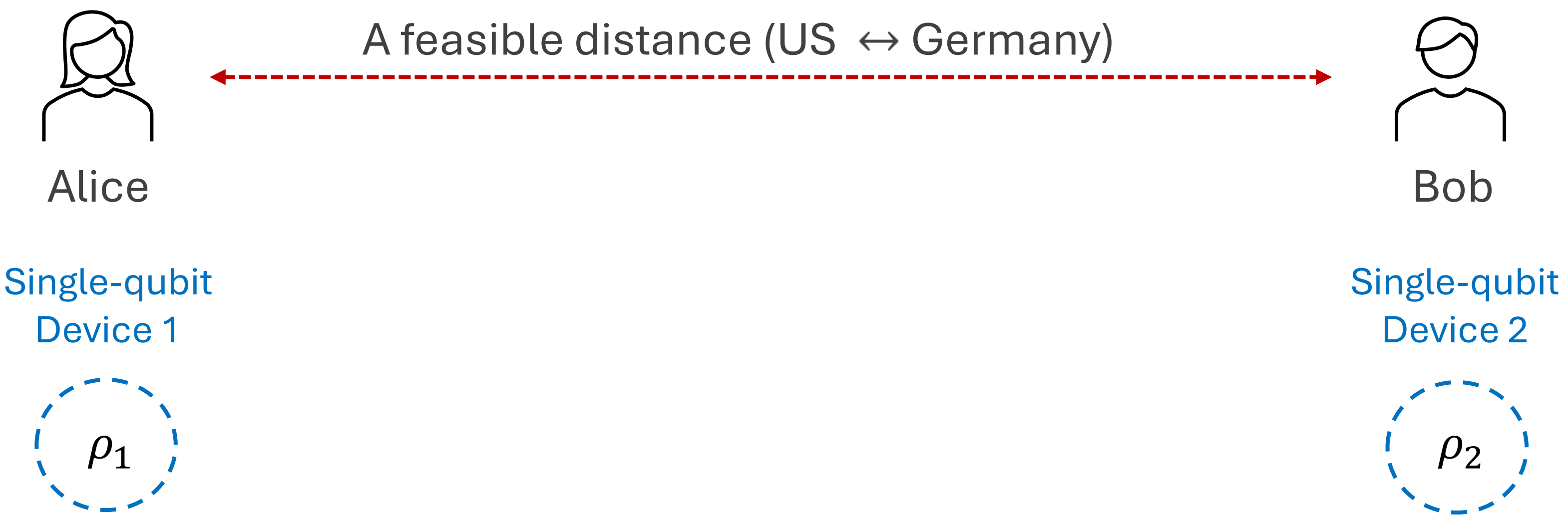


Measurement
in the Bell basis

The Bell basis:

$$\left\{ \begin{aligned} &M_{00}\colon |\beta_{00}\rangle\langle\beta_{00}|, \\ &M_{01}\colon |\beta_{01}\rangle\langle\beta_{01}|, \\ &M_{10}\colon |\beta_{10}\rangle\langle\beta_{10}|, \\ &M_{11}\colon |\beta_{11}\rangle\langle\beta_{11}|, \end{aligned} \right\}$$

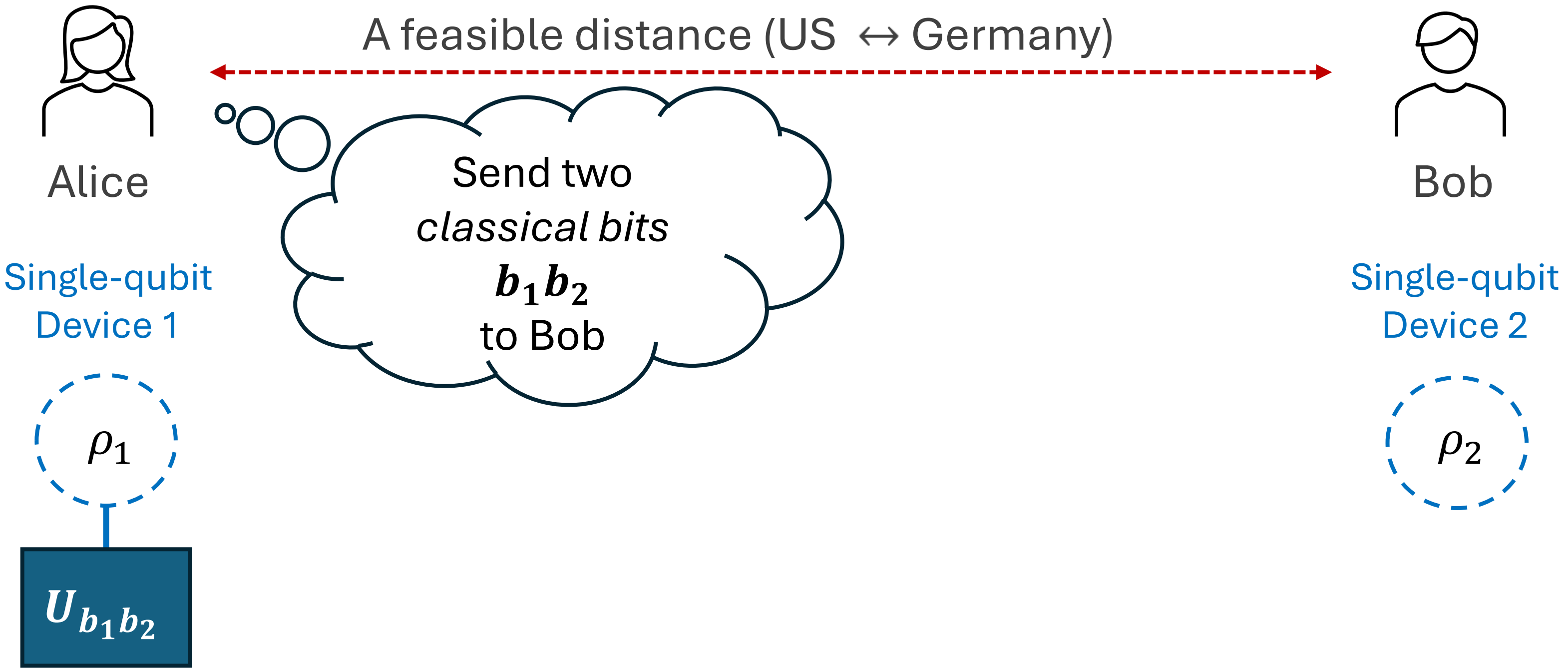
Superdense Coding



Total state:

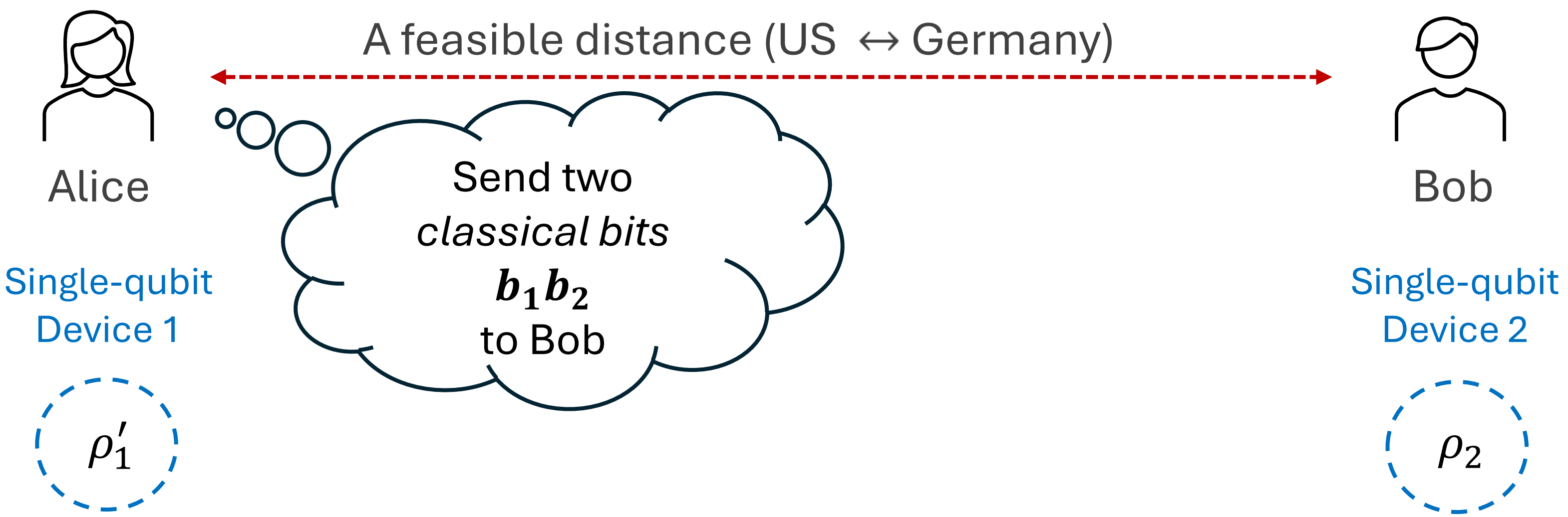
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Superdense Coding



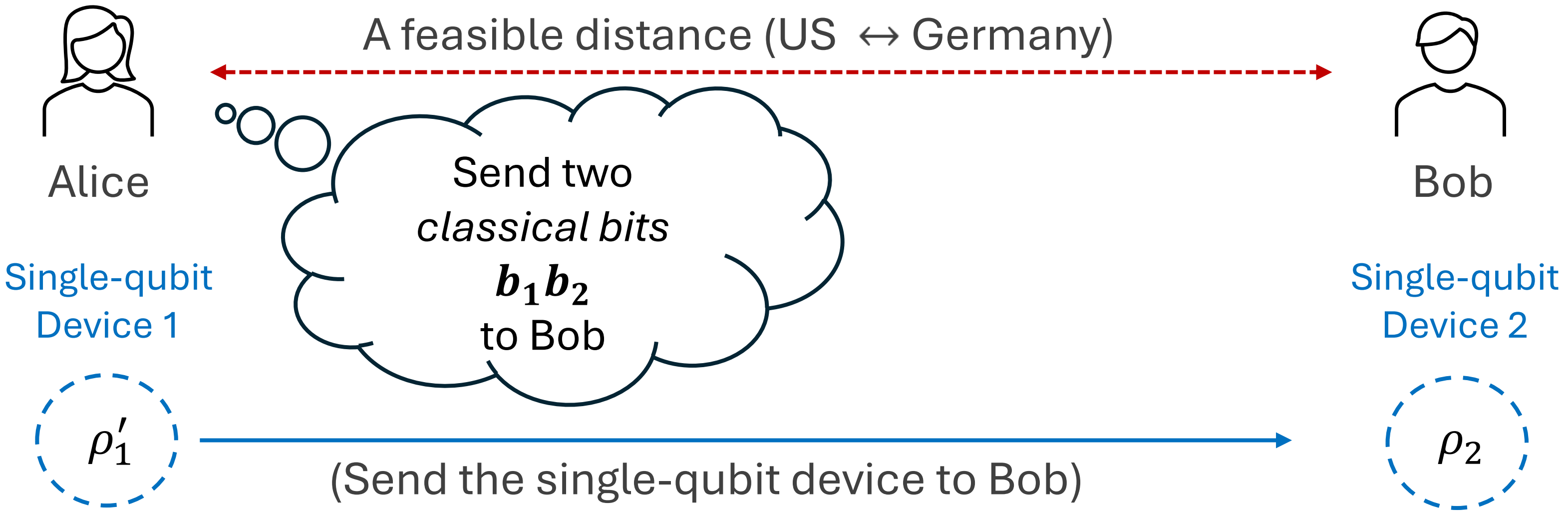
Total state
(before $\boldsymbol{U_{b_1 b_2}}$):
$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Superdense Coding



Total state (after $\boldsymbol{U_{b_1 b_2}}$): $|\beta_{b_1 b_2}\rangle$

Superdense Coding



Total state (after $\boldsymbol{U_{b_1 b_2}}$): $|\beta_{b_1 b_2}\rangle$

Superdense Coding

Alice

Send two *classical bits* $\boldsymbol{b_1 b_2}$ to Bob

Bob

Single-qubit Device 1

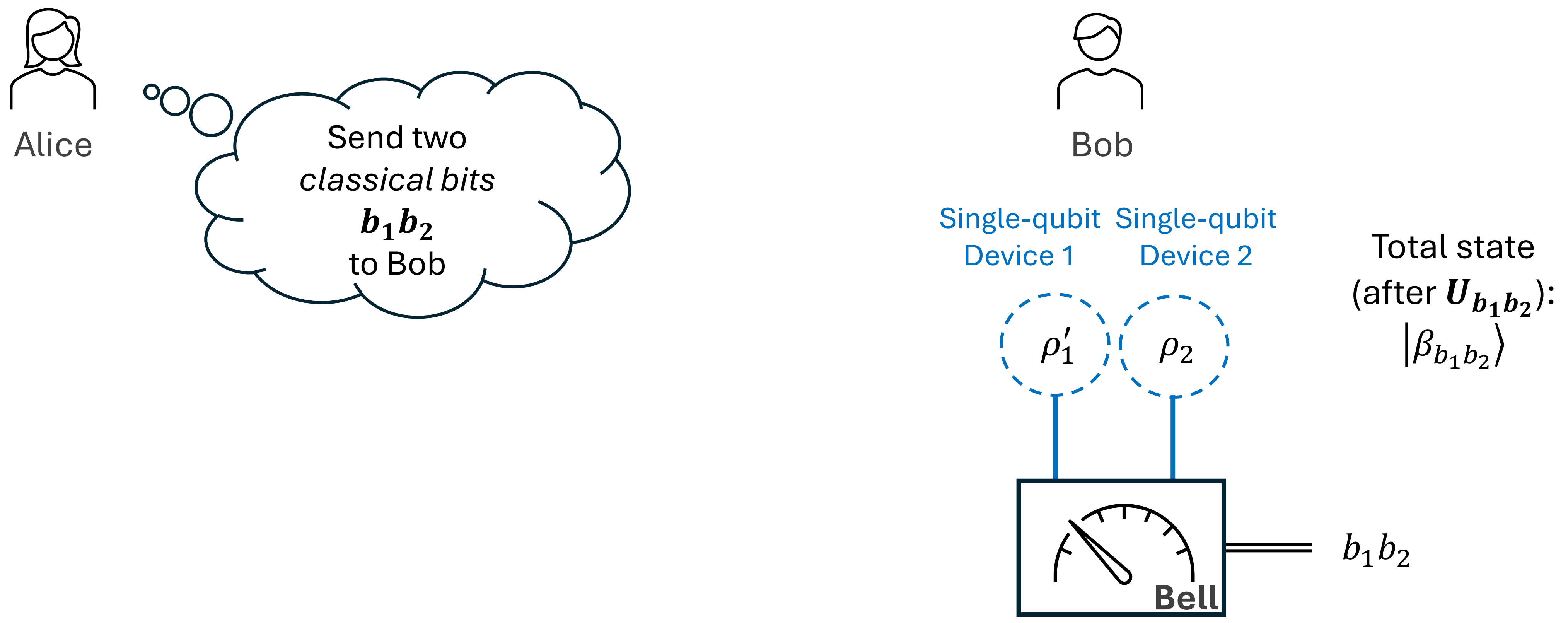
Single-qubit Device 2

ρ_1'

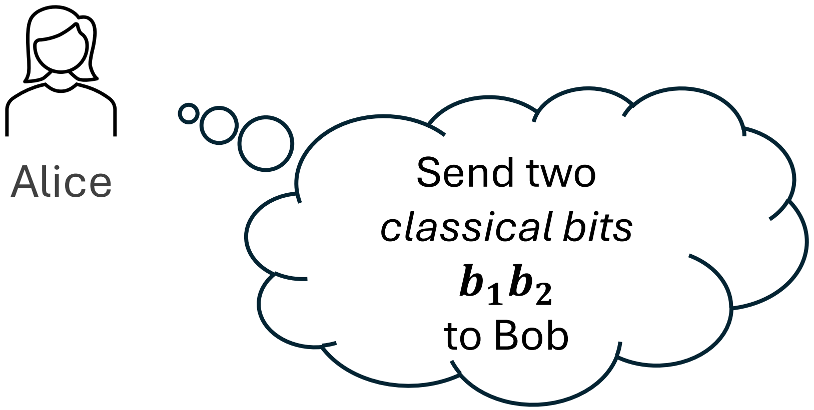
ρ_2

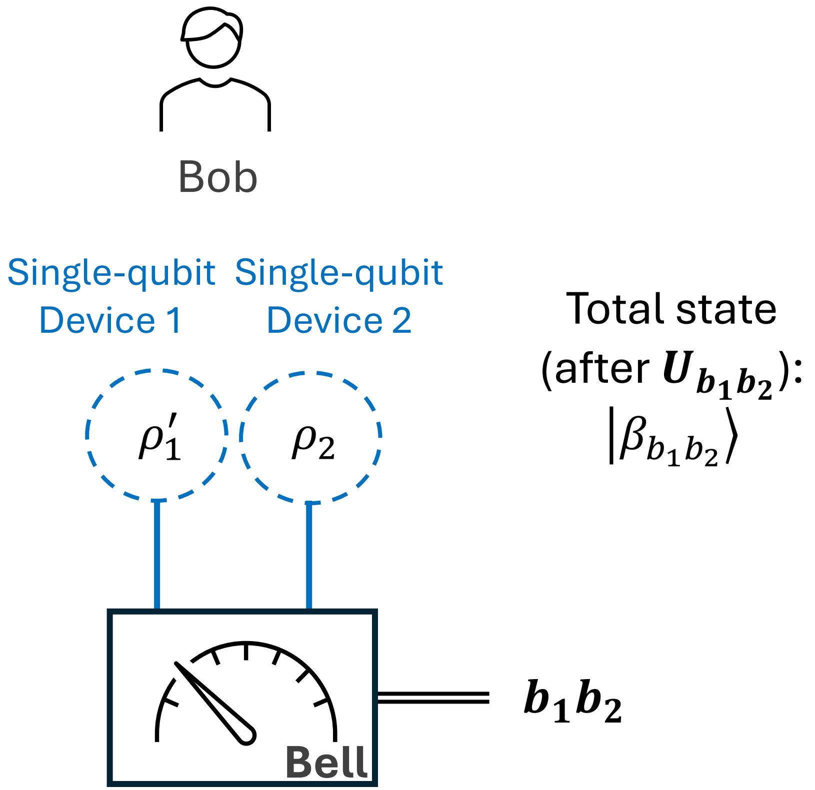
Total state (after $\boldsymbol{U_{b_1 b_2}}$):
$|\beta_{b_1 b_2}\rangle$

Superdense Coding



Superdense Coding

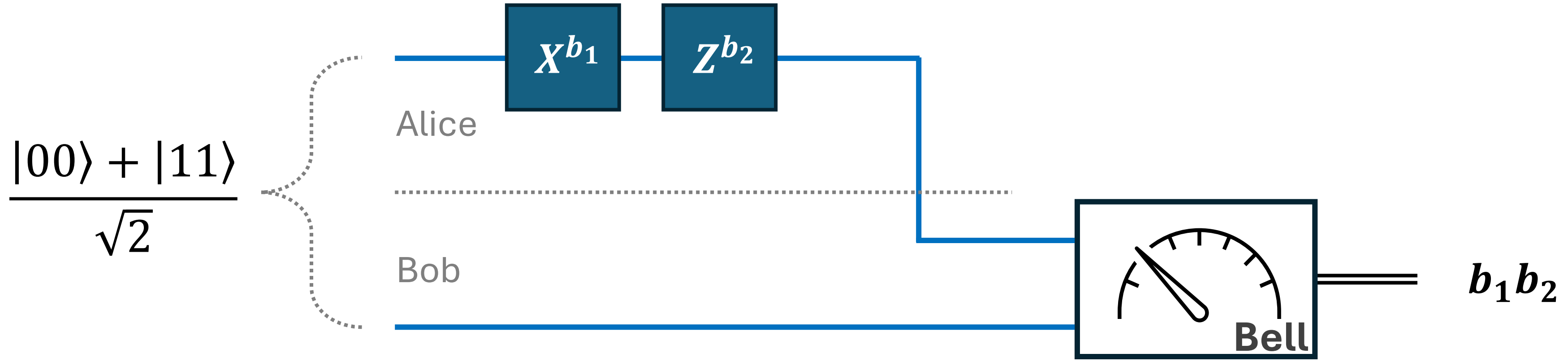




- **One** physical qubit can transmit **two** classical bits of information
  - (Require prior entanglement)
- **Superdense coding:** One qubit "encodes" two classical bits...

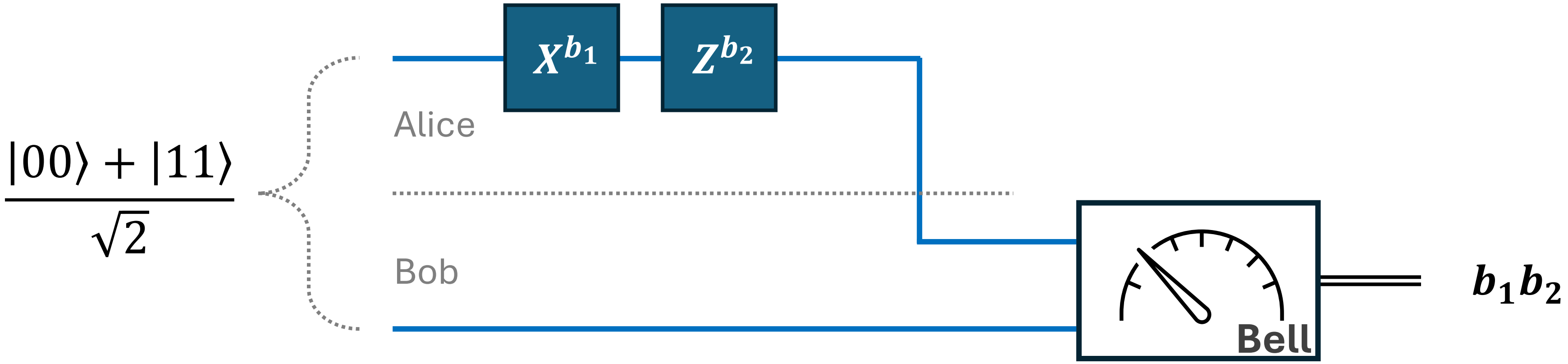
Superdense Coding

- A more compact description of the experiment:



Superdense Coding

- A more compact description of the experiment:



- Quick questions:
- (1) We wrote $U_{b_1 b_2} := Z^{b_2} X^{b_1}$ before, but why does X come first here?
- (2) How can we **transmit $2n$ classical bits using only n qubits?**

Quantum Teleportation

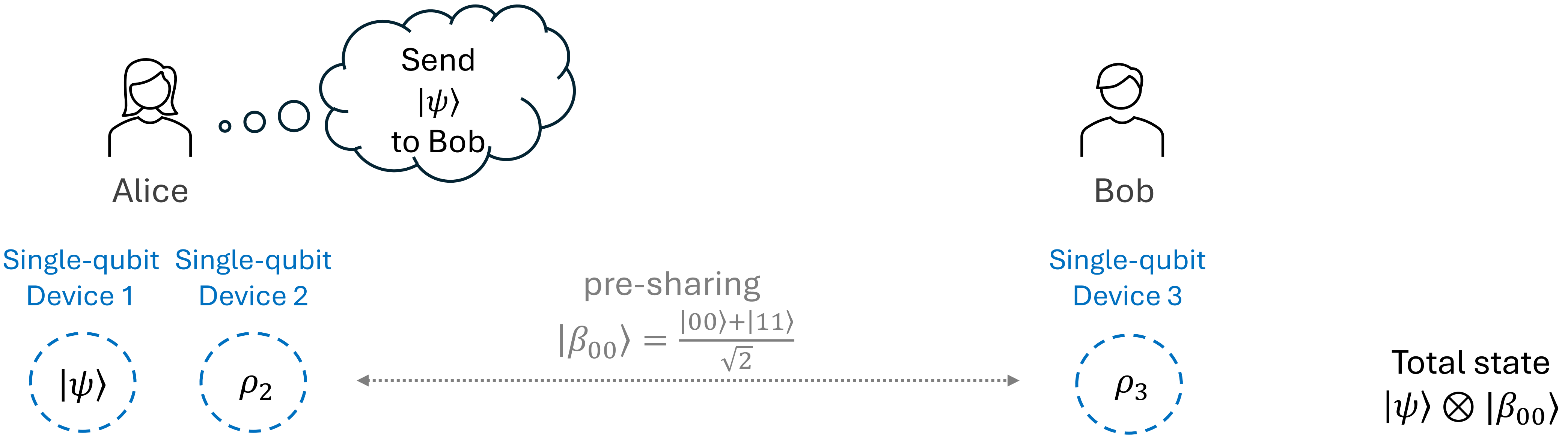
- Superdense coding: Transmit classical bits via qubits
- Quantum teleportation: **Transmit qubits via classical bits**

- A trivial approach (transportation): Physically deliver the device carrying $|\psi\rangle$ to Bob
- But here we consider **Teleportation**: Transmit via (quantum) channel

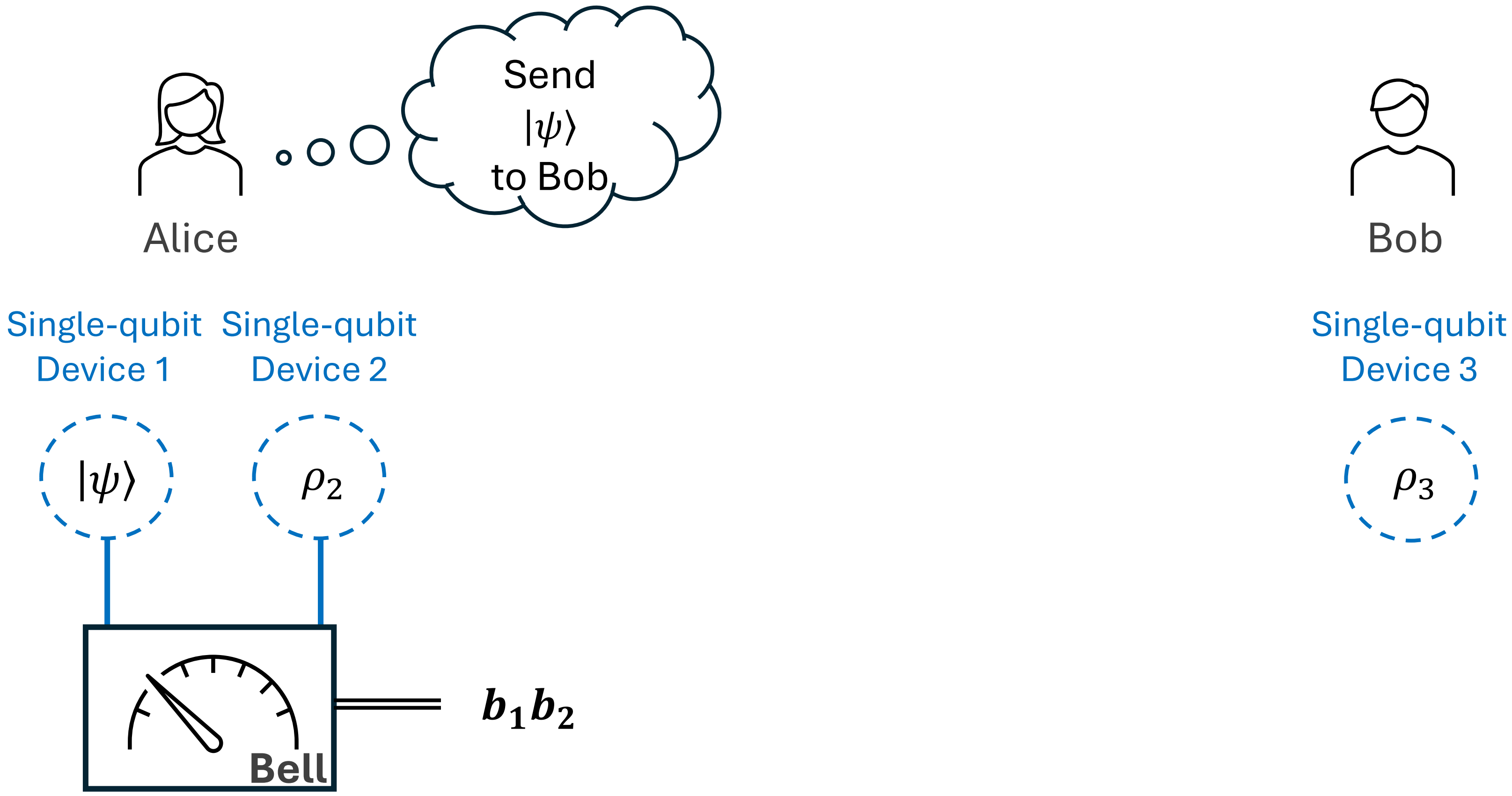
Quantum Teleportation

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Quantum Teleportation

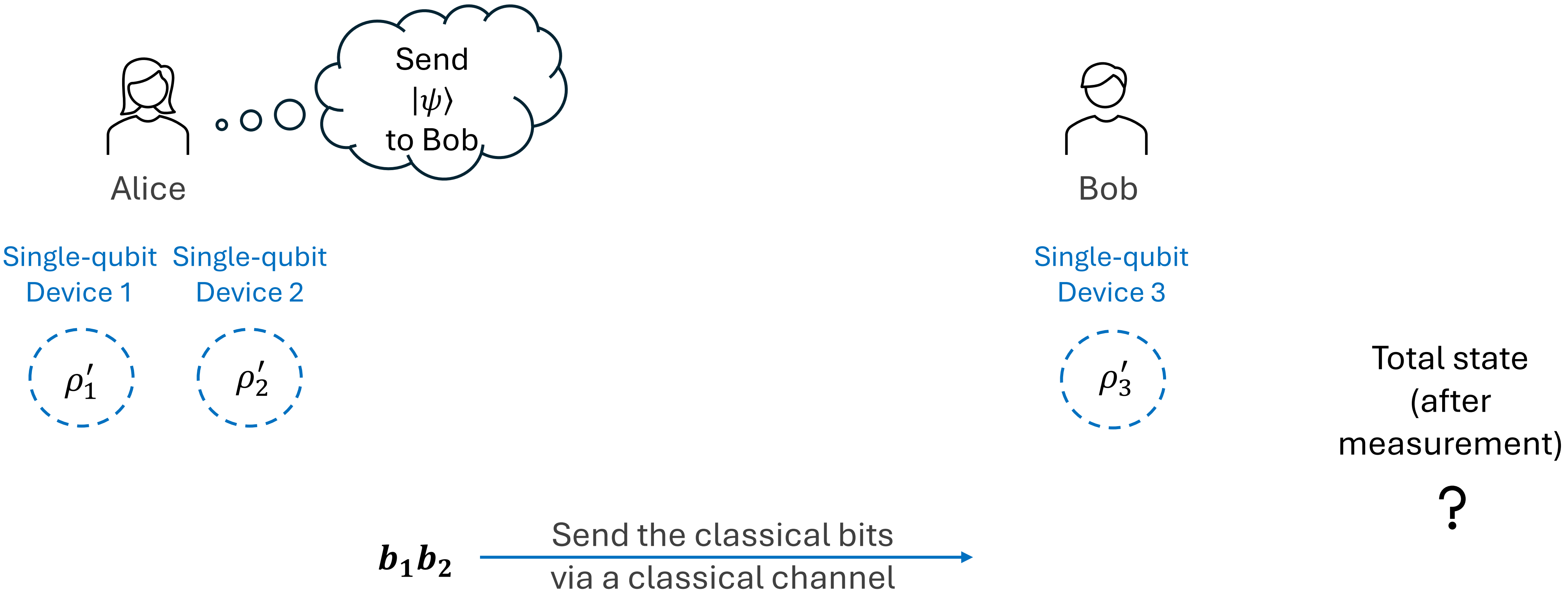


Quantum Teleportation

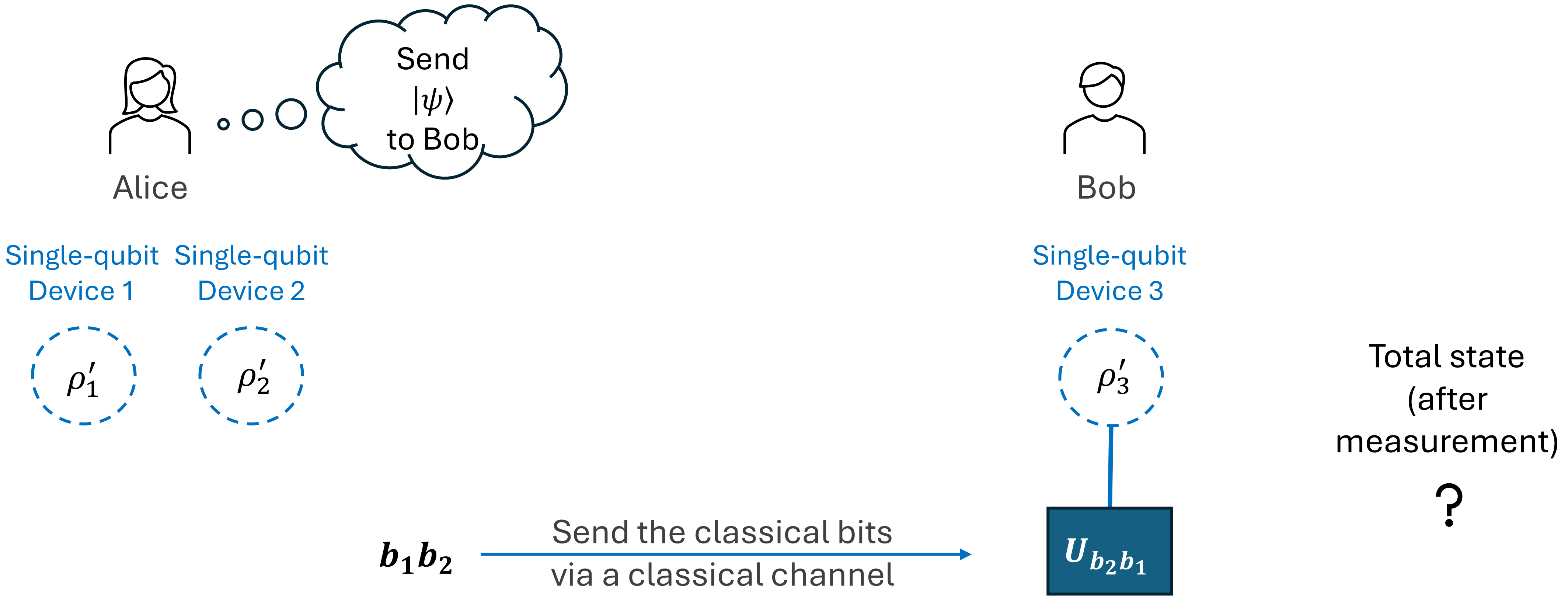


Total state (before measurement) $|\psi\rangle \otimes |\beta_{00}\rangle$

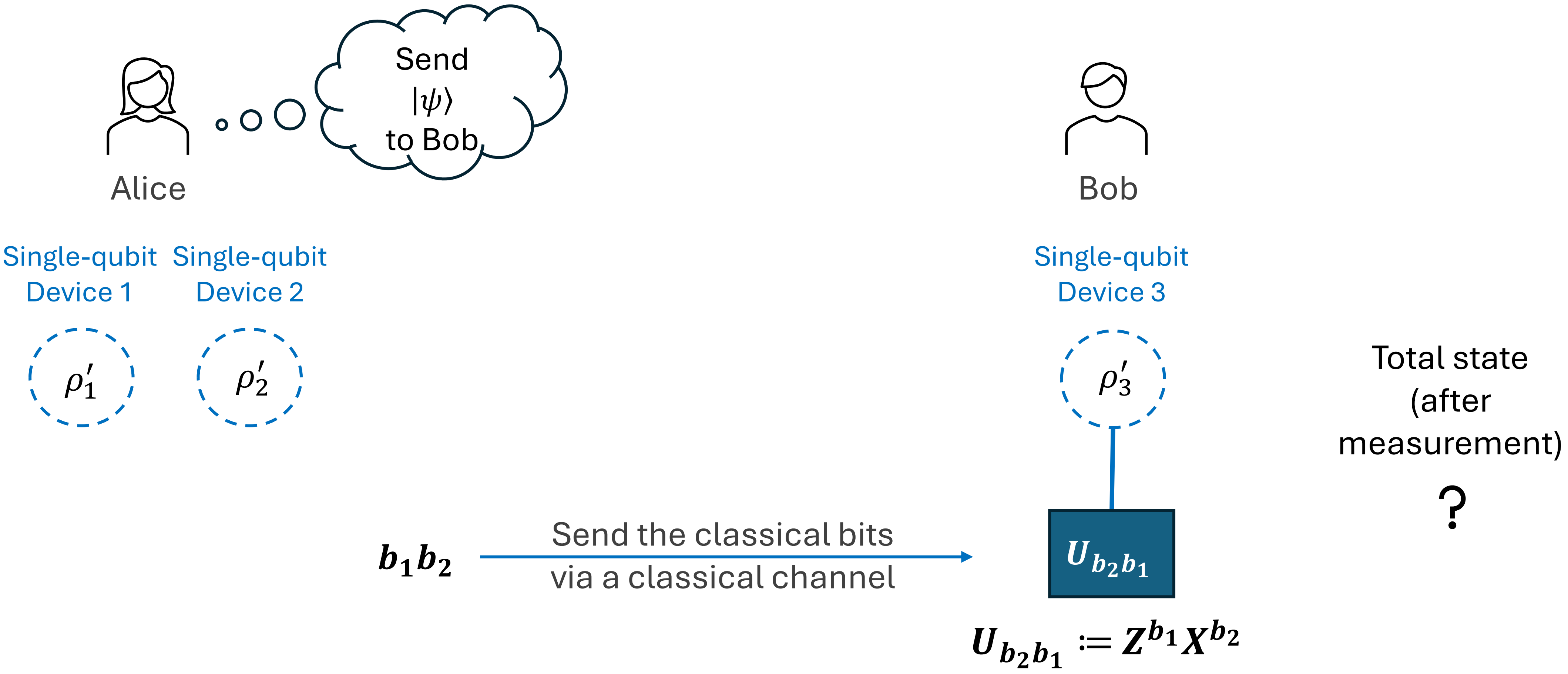
Quantum Teleportation



Quantum Teleportation

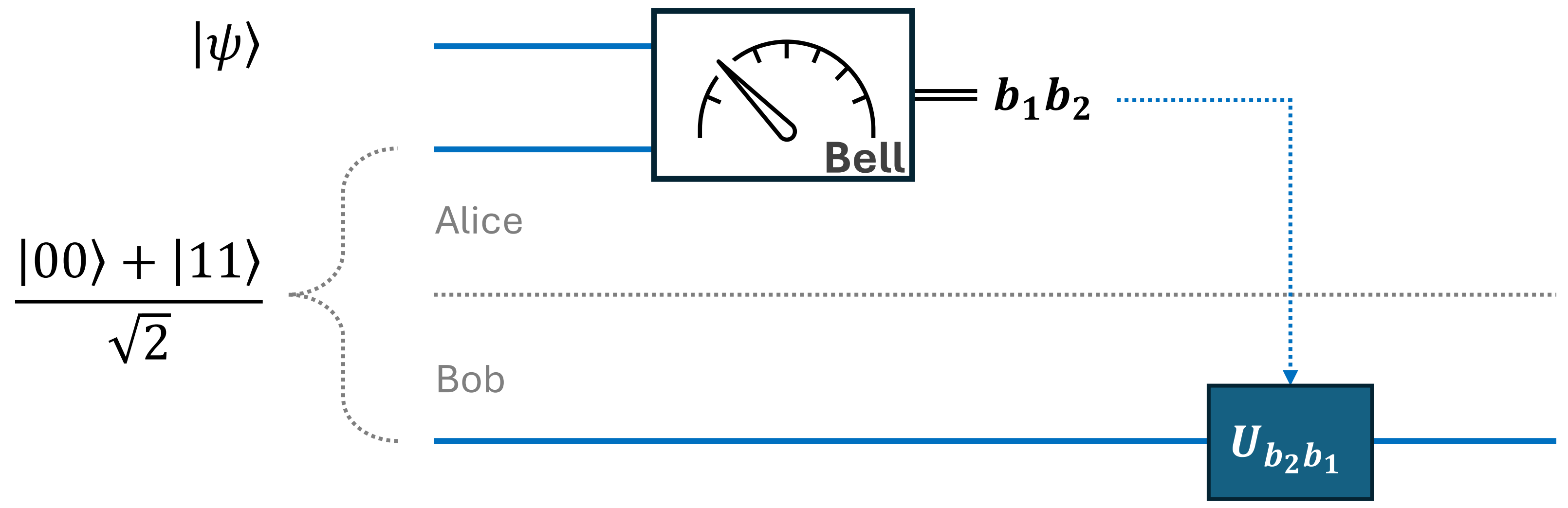


Quantum Teleportation



Quantum Teleportation

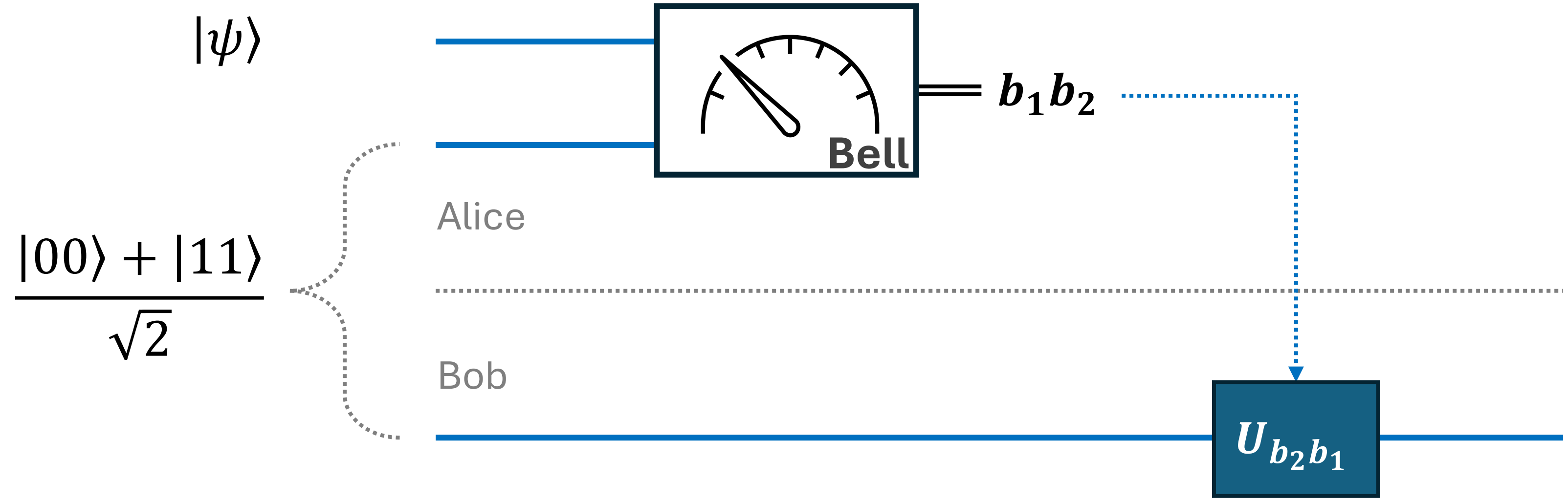
- **A more compact description (for analyzing states):**



Total states: $|\phi_0\rangle$ $|\phi_1\rangle$

Quantum Teleportation

- **A more compact description (for analyzing states):**

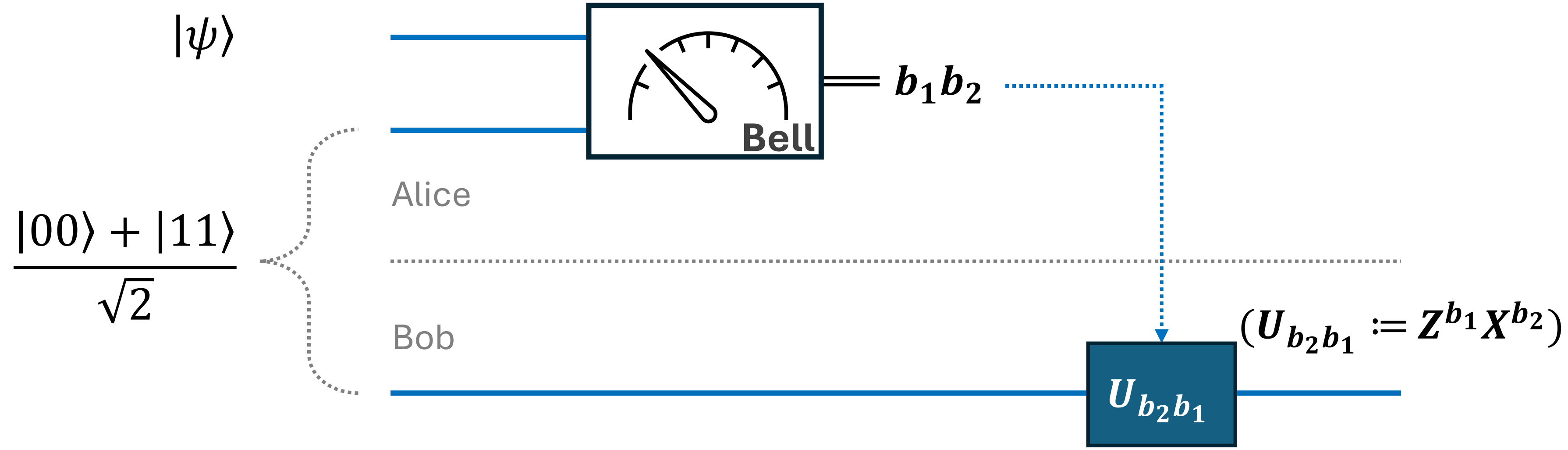


Total states: $|\phi_0\rangle$

Can we re-write the state of the first two systems using the Bell basis?

Quantum Teleportation

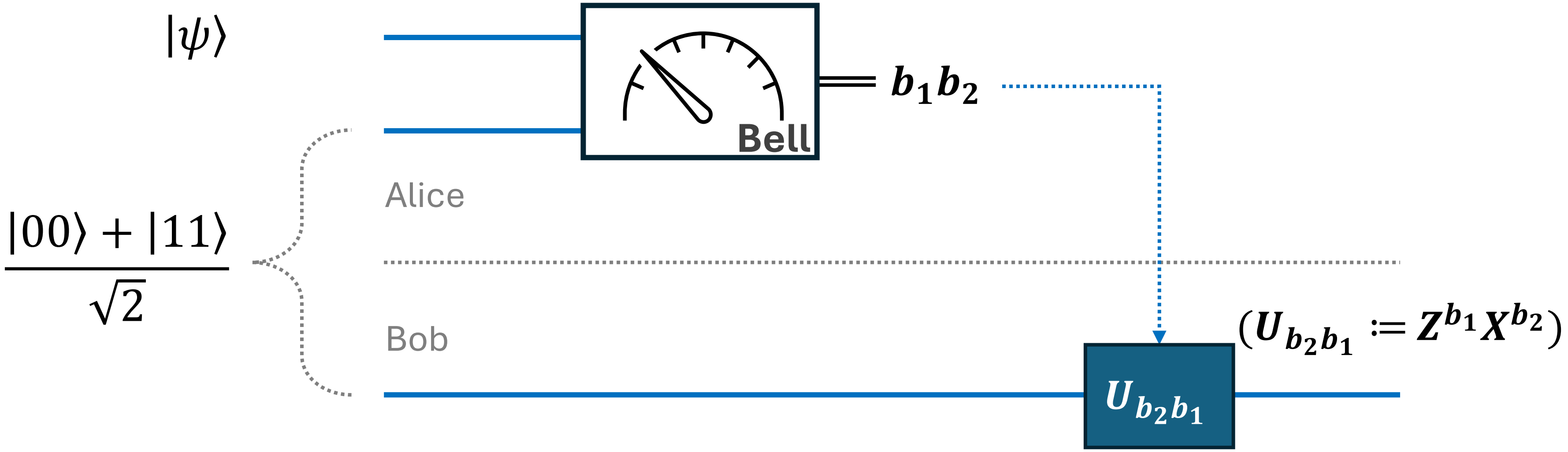
- **A more compact description (for analyzing states):**



Total states: $|\phi_0\rangle = \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle X^{b_2} Z^{b_1} |\psi\rangle$

Quantum Teleportation

- **A more compact description (for analyzing states):**



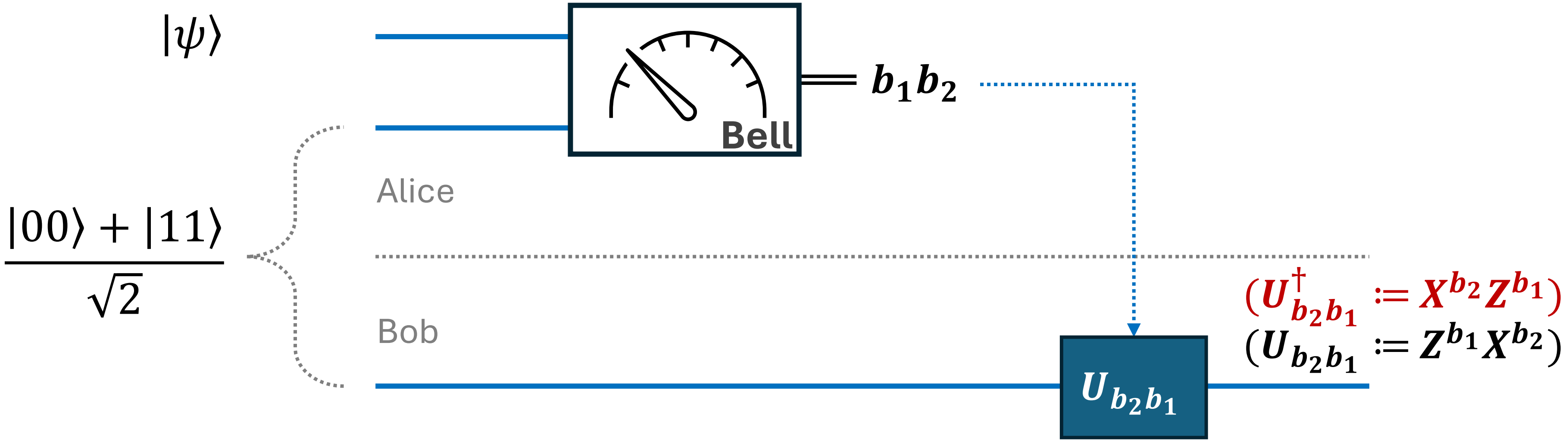
Total states:

$|\phi_0\rangle$
$= \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle \boldsymbol{X}^{b_2} \boldsymbol{Z}^{b_1} |\psi\rangle$

$|\phi_1\rangle$

Quantum Teleportation

- **A more compact description (for analyzing states):**



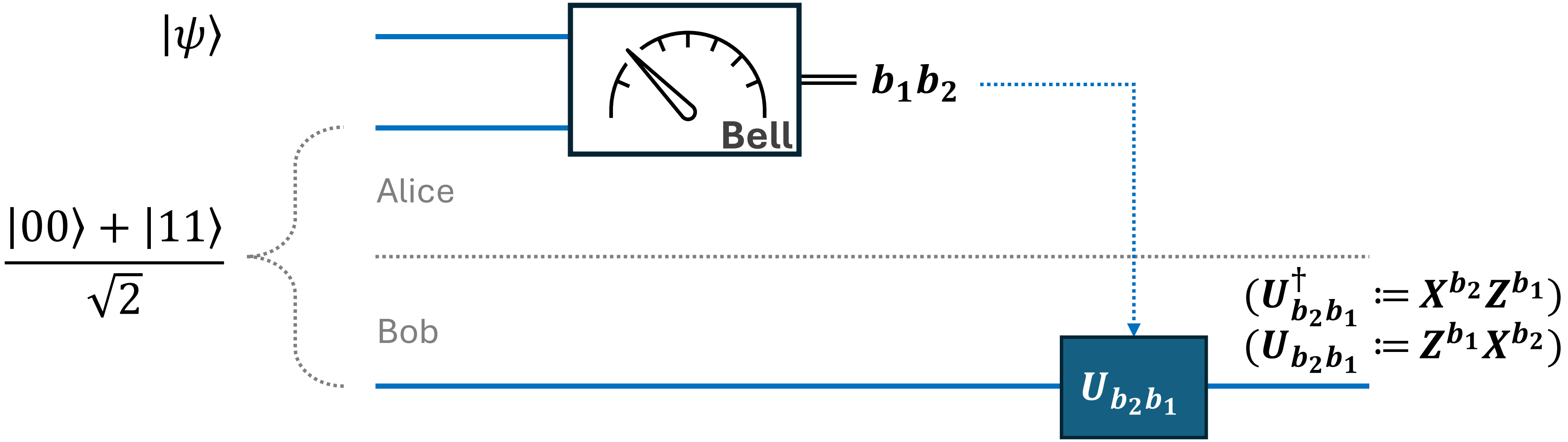
Total states:

$|\phi_0\rangle = \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle X^{b_2} Z^{b_1} |\psi\rangle$

$|\phi_1\rangle$

Quantum Teleportation

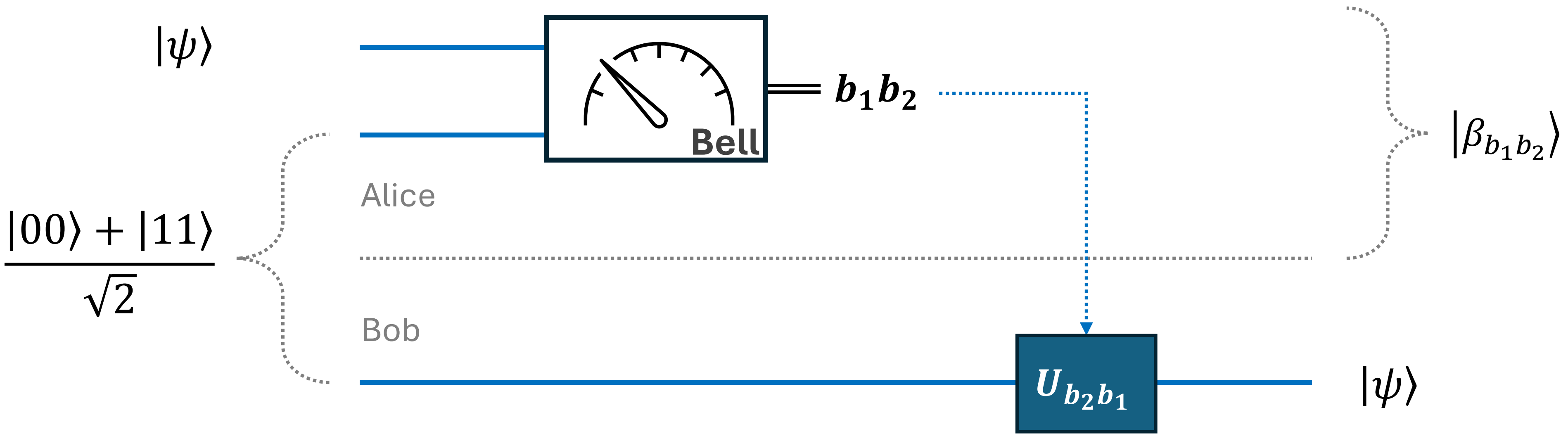
- **A more compact description (for analyzing states):**



Total states: $|\phi_0\rangle = \sum_{b_1,b_2\in\{0,1\}} |\beta_{b_1 b_2}\rangle U^\dagger_{b_2 b_1}|\psi\rangle$ $\quad |\phi_1\rangle = |\beta_{b_1 b_2}\rangle U^\dagger_{b_2 b_1}|\psi\rangle$ $\quad |\phi_2\rangle$

Quantum Teleportation

- **A more compact description (for analyzing states):**



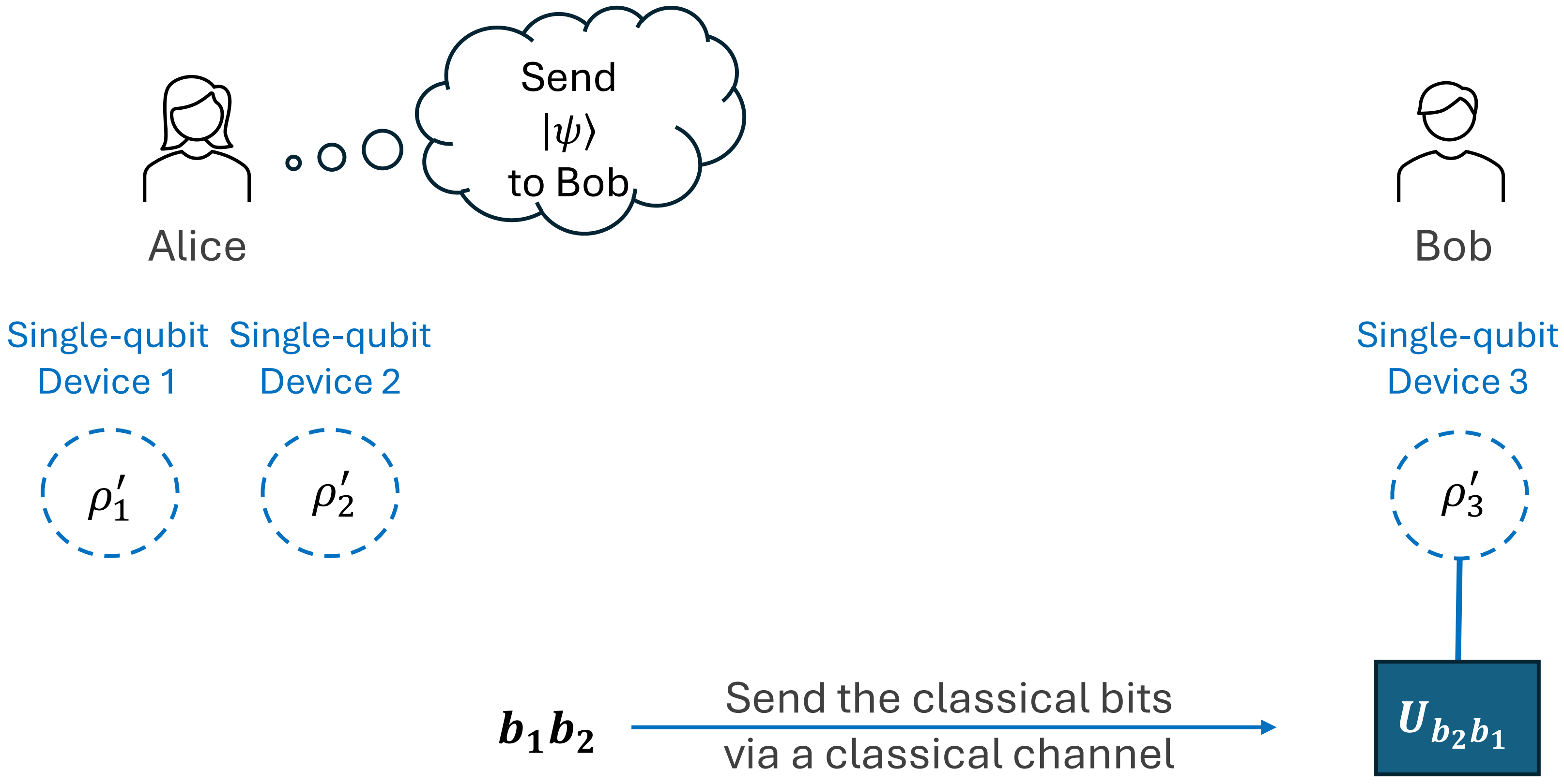
Total states:

$|\phi_0\rangle = \sum_{b_1,b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle \boldsymbol{U}^{\dagger}_{\boldsymbol{b_2 b_1}} |\psi\rangle$

$|\phi_1\rangle = |\beta_{b_1 b_2}\rangle \boldsymbol{U}^{\dagger}_{\boldsymbol{b_2 b_1}} |\psi\rangle$

$|\phi_2\rangle = |\beta_{b_1 b_2}\rangle \otimes |\psi\rangle$

Quantum Teleportation



Total state (after measurement):

$$|\boldsymbol{\beta}_{\boldsymbol{b_1 b_2}}\rangle \otimes \boldsymbol{U}^{\dagger}_{\boldsymbol{b_1 b_2}} |\boldsymbol{\psi}\rangle$$

Quantum Teleportation

Alice

Send
$|\psi\rangle$
to Bob

Bob

Single-qubit Device 1

Single-qubit Device 2

Single-qubit Device 3

ρ_1' ρ_2'

$|\psi\rangle$

$\underbrace{\rho_1' \quad \rho_2'}_{|\beta_{b_1 b_2}\rangle}$

Total state (Final):

$$|\boldsymbol{\beta}_{\boldsymbol{b_1 b_2}}\rangle \otimes |\boldsymbol{\psi}\rangle$$

Reference

- **[NC00]:** Sections 1.3.6 and 1.3.7
- **[KLM07]:** Chapter 5

Next topic

- **Quantum Circuits**
  - **Controlled operations and Measurement**