Exercises – Weeks 2 and 3

Exercise 1: Guassian elimination

(1) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Solve the linear system $\mathbf{A}\vec{x} = \vec{b}$ using Gaussian elimination.

(2) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \\ -1 & -2 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 6 \\ -5 \end{bmatrix}$. Solve the linear system $\mathbf{A}\vec{x} = \vec{b}$ using Gaussian elimination.

Exercise 2: Linear dependence

(1) Show $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are linearly dependent.

(2) Show $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly independent.

(3) Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be vectors in $\mathbb{C}^n$, where $m > n$. Prove that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ **can never be** linearly independent.

Exercise 3: Matrix multiplication and linear combination

Let $\mathbf{A} = [\vec{a}_1, \cdots, \vec{a}_m]$ be an arbitrary matrix and $\vec{a}_i$ be its i-th column vector. Prove that, for any vector $\vec{x}^T = [x_1, \cdots, x_m]$, we can write $\mathbf{A}\vec{x} = \sum_{i=1}^{m} x_i \vec{a}_i$.

Suppose that $\mathbf{A}$ is an $n \times m$ matrix where $n > m$. Prove that $\mathsf{rank}(\mathbf{A}) = m$ if and only if $\mathbf{A}\vec{x} = 0$ has only one solution (and what is it?).

Exercise 4: Qubits

Determine whether the following vectors represent valid qubits (i.e., unit vectors in $\mathbb{C}^2$):

(1) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (2) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (3) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (4) $\frac{1}{3}|0\rangle + \frac{2}{3}|1\rangle$ (5) $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|+\rangle$ (6) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$

Exercise 5: Inner product

(1) Let $\vec{x}$ be a $n \times 1$ vector. Show that $\vec{x} = \sum_{i=0}^{n-1} x_i \cdot |i\rangle$.

(2) Let $\mathbf{A}$ be a 2×2 matrix. What are $\langle 0|\mathbf{A}|0\rangle$ and $\langle 1|\mathbf{A}|1\rangle$?

(3) More generally, if $\mathbf{A}$ is an $2^n \times 2^n$ matrix, what is $\langle i|\mathbf{A}|i\rangle$ for $(1 \le i \le n)$?

Exercise 6: Matrix of the NOT transformation

Design a matrix $\mathbf{A}$ such that $\mathbf{A}|0\rangle = |1\rangle$ and $\mathbf{A}|1\rangle = |0\rangle$.

Exercise 7: Hadamard matrix

Let $\mathbf{H} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Show that (1) $\mathbf{H}^2 = \mathbf{I}$, and

(2) $\mathbf{H}|b\rangle = \frac{|0\rangle + (-1)^b |1\rangle}{\sqrt{2}} (b \in \{0, 1\})$

Exercise 8: Tensor Product - 1 [NC00, Exercise 2.28]

Prove that (1)$(\mathbf{A} \otimes \mathbf{B})^* = \mathbf{A}^* \otimes \mathbf{B}^*$, (2) $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$, (3) $(\mathbf{A} \otimes \mathbf{B})^\dagger = \mathbf{A}^\dagger \otimes \mathbf{B}^\dagger$, and (4)$\exists \mathbf{A}, \mathbf{B}$ s.t. $(\mathbf{A} \otimes \mathbf{B}) \neq (\mathbf{B} \otimes \mathbf{A})$.

Exercise 9: Tensor Product - 2

Let $|\phi\rangle := \begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}$ and $|\psi\rangle := \begin{bmatrix}\alpha_2\\ \beta_2\end{bmatrix}$. Compute

$$|\phi\rangle\langle\phi| \otimes |\psi\rangle\langle\psi| \text{ and } |\phi\rangle\langle\phi| + |\psi\rangle\langle\psi|.$$

What if $|\phi\rangle = |0\rangle$ and $|\psi\rangle = |1\rangle$? Prove that $\mathbf{I}_{n\times n} = \sum_{i=0}^{n-1} |i\rangle\langle i|$.

Exercise 10: Tensor Product - 3 (Hard, you may need to understand the block representation of matrix multiplication)

Prove that $(\mathbf{A}\vec{x}) \otimes (\mathbf{B}\vec{y}) = (\mathbf{A} \otimes \mathbf{B})(\vec{x} \otimes \vec{y})$ (suppose that their sizes match)

Exercise 11: Global phase

Let $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ be a single-qubit state. Show that $|\phi\rangle$ can be written as $|\phi\rangle = e^{i\gamma}(\cos\frac{\theta}{2} \cdot |0\rangle + e^{i\psi}\sin\frac{\theta}{2} \cdot |0\rangle)$, where $\gamma, \theta, \psi \in \mathbb{R}$.

Exercise 12: Measurement and Bases

Let $|+\rangle := \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle := \frac{|0\rangle-|1\rangle}{\sqrt{2}}$ be two quantum states.

1. Prove that $|+\rangle$ and $|-\rangle$ form an orthonormal basis of $\mathbb{C}^2$
2. Let $|\phi\rangle := \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$ be a quantum state. What's the outcome distribution if I measure $|\phi\rangle$ in the basis $\{|+\rangle, |-\rangle\}$

Exercise 13: An example of entanglement

Let $|\psi\rangle := \frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

1. What is the vector representation of $|\psi\rangle$?
2. Is it possible to write
$$|\psi\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)?$$
for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$?