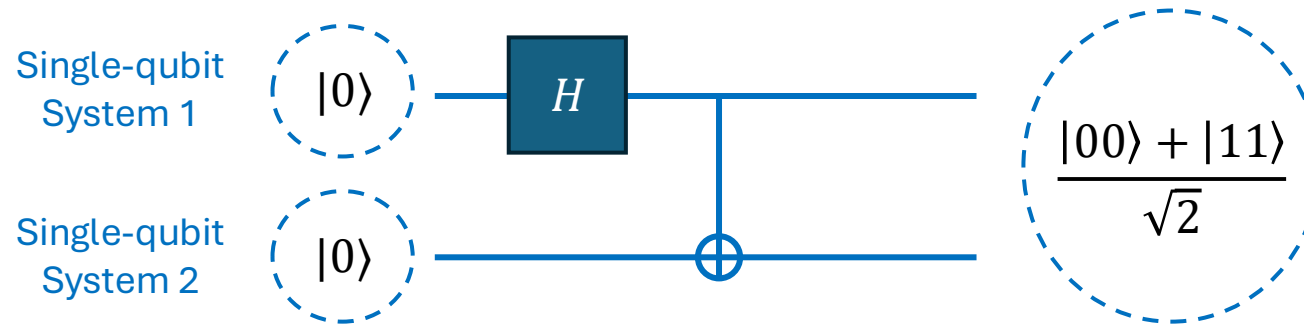


# Quantum Computing

- Week 13 (July 7-8, 2026)
- Topics:
  - Density operator

# Mixed States, Recaps

- **Pure state:** Can be described by a state vector
- **Mixed state:** Cannot ...



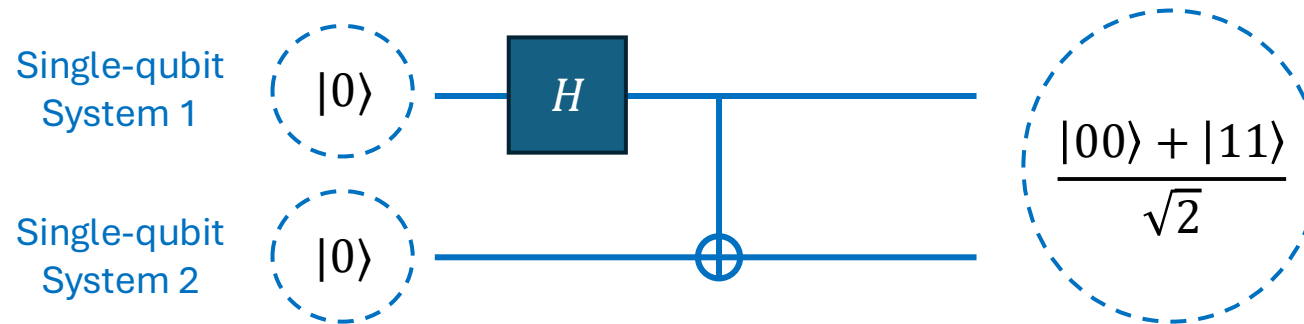
## Small Exercise: (pure or mixed)

1. The initial state of system 1 is \_\_\_\_.
2. The states of systems 1 and 2 (after H and CNOT) are both \_\_\_\_.
3. The state of the total system (after H and CNOT) is \_\_\_\_.

# Mixed States, Recaps

- **Pure state:** Can be described by a state vector
- **Mixed state:** Cannot ...

How can we describe mixed states?



## Small Exercise: (pure or mixed)

1. The initial state of system 1 is pure.
2. The states of systems 1 and 2 (after H and CNOT) are both mixed.
3. The state of the total system (after H and CNOT) is pure.

# Density Operator

- Let  $\{|\psi_i\rangle\}_i$  be a set of pure states, where  $i$  is an index
- Suppose that a quantum system is in  $|\psi_i\rangle$  with probability  $p_i$  (s.t.  $\sum_i p_i = 1$ )
- Then we write the **density operator**  $\rho$  of the system as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

- Examples:
  - A single-qubit system with state  $|0\rangle$
  - A single-qubit system with state  $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$
  - A single-qubit system that is in state  $|0\rangle$  with probability  $\frac{1}{2}$  and in state  $|1\rangle$  with probability  $\frac{1}{2}$

# Density Operator

- $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$  (where  $\sum_i p_i = 1$ )

**What's the difference between the states of the two systems?**

**System 1:** Sample two bits  $b_1 b_2$  uniformly at random, and set its state as  $|b_1 b_2\rangle$

**System 2:**  $\frac{|00\rangle+|01\rangle+|10\rangle+|11\rangle}{2}$

# Density Operator

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**System 2:**  $\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$

$$\rho_1 = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho_2 = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

# Density Operator

- $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$  (where  $\sum_i p_i = 1$ )
- Density operator **provides another way to formulate postulates** of quantum computing:

Postulate	State vector	Density operator
Describing the state of a system	$ \psi\rangle = \sum_i \alpha_i  \psi_i\rangle$	$\rho = \sum_i p_i  \psi_i\rangle\langle\psi_i $
Unitary transformation	$ \psi\rangle \mapsto U \psi\rangle$	$\rho \mapsto U\rho U^\dagger$
Quantum measurement $\{M_m\}_m$	$p(m) = \langle\psi M_m^\dagger M_m \psi\rangle,$ $ \psi_m\rangle = \frac{M_m}{\sqrt{p(m)}}  \psi\rangle$	$p(m) = \text{tr}(M_m^\dagger M_m \rho),$ $\rho_m = \frac{M_m \rho M_m^\dagger}{p(m)}$
Composite system	$ \psi_1\rangle \otimes \dots \otimes  \psi_n\rangle$	$\rho_1 \otimes \dots \otimes \rho_n$

# Trace

- The sum of diagonal elements:

$$\text{tr}(M) = \sum_i M_{ii}$$

- Properties of trace:

- **Linearity:**  $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ ,  $\text{tr}(z \cdot A) = z \cdot \text{tr}(A)$
- **Cyclicity:**  $\text{tr}(AB) = \text{tr}(BA)$  (similarly,  $\text{tr}(ABC) = \text{tr}(CAB) = \text{tr}(BCA)\dots$ )

- Several facts implied by **Cyclicity:**

- $\text{tr}(U\rho U^\dagger) = \text{tr}(\rho)$
- $\text{tr}(M|\psi\rangle\langle\psi|) = \text{tr}(\langle\psi|M|\psi\rangle) = \langle\psi|M|\psi\rangle$

# Trace

- The **sum of diagonal elements**:  $\text{tr}(M) = \sum_i M_{ii}$
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  - $\text{tr}(U\rho U^\dagger) = \text{tr}(\rho)$
  - $\text{tr}(M|\psi\rangle\langle\psi|) = \text{tr}(\langle\psi|M|\psi\rangle) = \langle\psi|M|\psi\rangle$ , (and thus  $\text{tr}(|\psi\rangle\langle\psi|) = \langle\psi|\psi\rangle = 1$ )
- Some examples:
  - Unitary transformation formulated by state vectors v.s. by density operators
  - Measurement formulated by state vectors v.s. by density operators

# Trace

- **Criterion** to decide if a state is mixed or pure: Let  $\rho$  be the density operator of a quantum system.
  - $\text{tr}(\rho^2) < 1$ : Mixed state
  - $\text{tr}(\rho^2) = 1$ : Pure state

# Partial Trace and Reduced Density Operator

- Let:
  - $A$  and  $B$  be two quantum systems
  - Let  $|a_1\rangle$  and  $|a_2\rangle$  be any two state vectors defined over (the state space of)  $A$
  - Let  $|b_1\rangle$  and  $|b_2\rangle$  be any two state vectors defined over  $B$

- **Partial Trace:**

$$\mathit{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) := |a_1\rangle\langle a_2| \cdot \mathit{tr}(|b_1\rangle\langle b_2|)$$

- Remark:
  1.  $\mathit{tr}$  maps an operator onto a complex number, but  $\mathit{tr}_B$  maps an operator onto an operator.
  2. The operator getting from  $\mathit{tr}_B$  is defined over  $A$

# Partial Trace and Reduced Density Operator

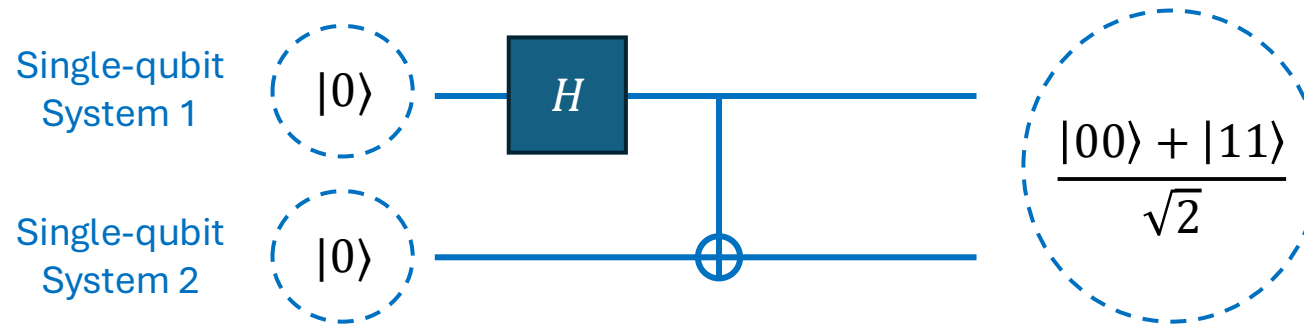
- Let:
  - $A$  and  $B$  be two quantum systems
  - Let  $\rho_{AB}$  be the state of the state space of the composite system  $A \otimes B$
- **Reduced Density Operator of  $A$  :**

$$\rho_A := \text{tr}_B(\rho_{AB})$$

- Examples (of calculating  $\rho_A$ ) :
  1.  $\rho_{AB} = \rho \otimes \sigma$ , where  $\rho$  is defined over  $A$  and  $\sigma$  is defined over  $B$
  2.  $\rho_{AB} = |00\rangle\langle 00| (= |0\rangle\langle 0| \otimes |0\rangle\langle 0|)$
  3.  $\rho_{AB} = \left( \left( \frac{|0\rangle+|1\rangle}{\sqrt{2}} \right) \otimes \left( \frac{|0\rangle-|1\rangle}{\sqrt{2}} \right) \right) \left( \left( \frac{\langle 0|+\langle 1|}{\sqrt{2}} \right) \otimes \left( \frac{\langle 0|-\langle 1|}{\sqrt{2}} \right) \right)$

# Partial Trace and Reduced Density Operator

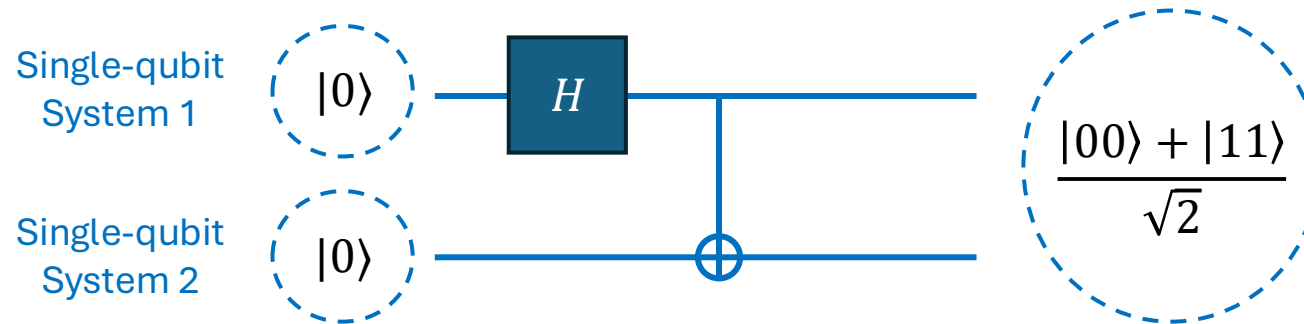
- The final state is  $\rho_{12} = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$



- $\rho_1 = \text{tr}_2(\rho_{12}) = ?$

# Partial Trace and Reduced Density Operator

- The final state is  $\rho_{12} = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)$



- $\rho_1 = \text{tr}_2(\rho_{12}) = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

# Pure state and Mixed state

- $\rho = |\psi\rangle\langle\psi|$ , where  $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

- $\rho' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

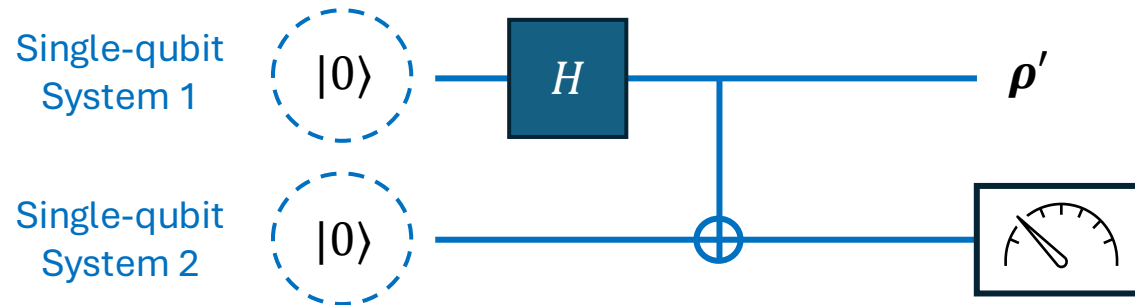
# Pure state and Mixed state

- $\rho = |\psi\rangle\langle\psi|$ , where  $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ 
  - $\text{tr}(\rho^2) = 1 \Rightarrow$  A pure state.
  - We “somehow” know the state of the system with **certainty** (i.e., the state is  $|\psi\rangle$  with probability 1)
  - A system is in pure state means the system can be described by one state vector
  
- $\rho' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$

# Pure state and Mixed state

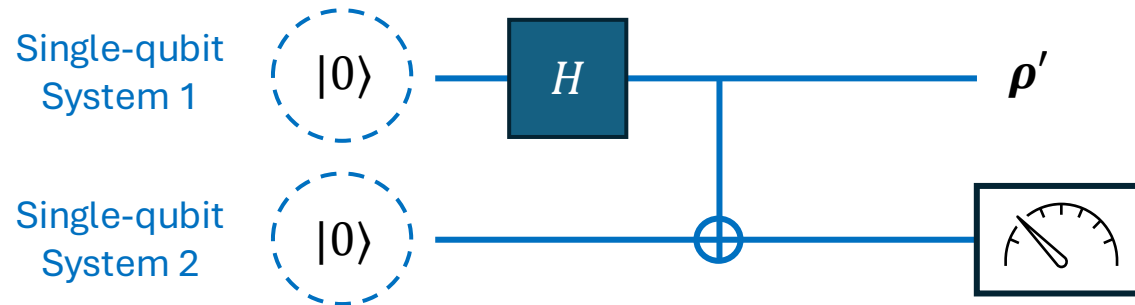
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- $\rho' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ 
  - $\text{tr}(\rho'^2) = \frac{1}{2} < 1 \Rightarrow$  A mixed state
  - We are **uncertain** about the exact state the system is in...

# Pure state and Mixed state



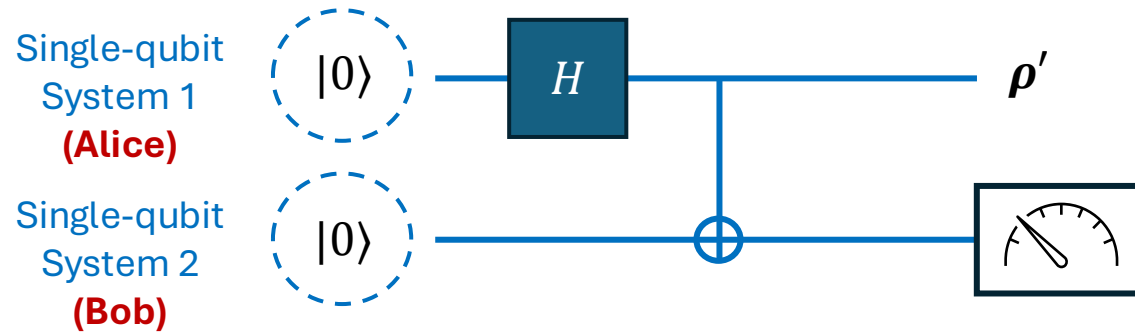
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  - The measurement (on system 2) outcome is a uniformly random bit, corresponding to the state and the probability distribution of system 1...

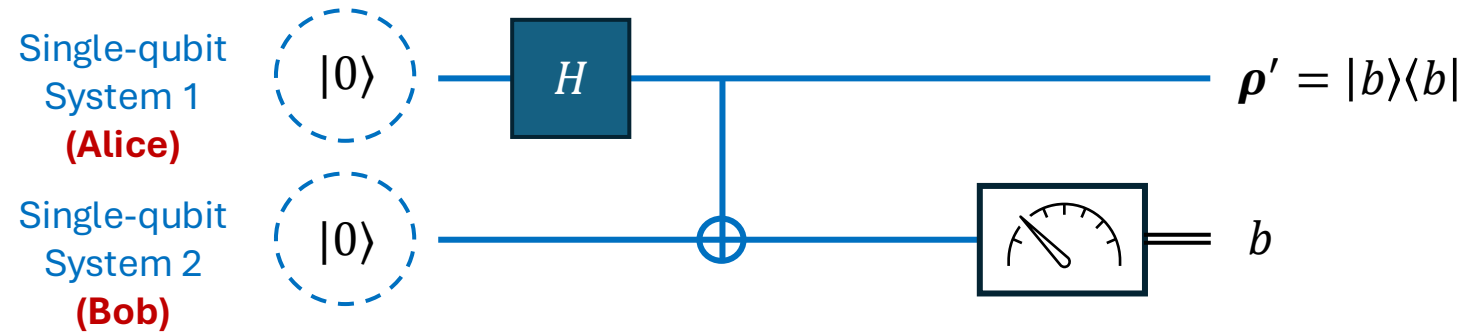
# Pure state and Mixed state



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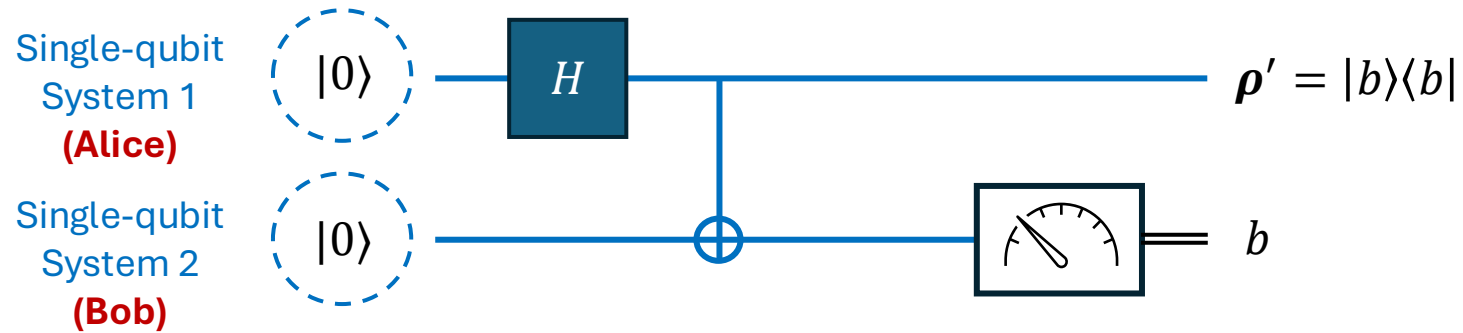
What if Bob tells the measurement outcome to Alice?

# Pure state and Mixed state



- $\rho' = |b\rangle\langle b|$

# Pure state and Mixed state

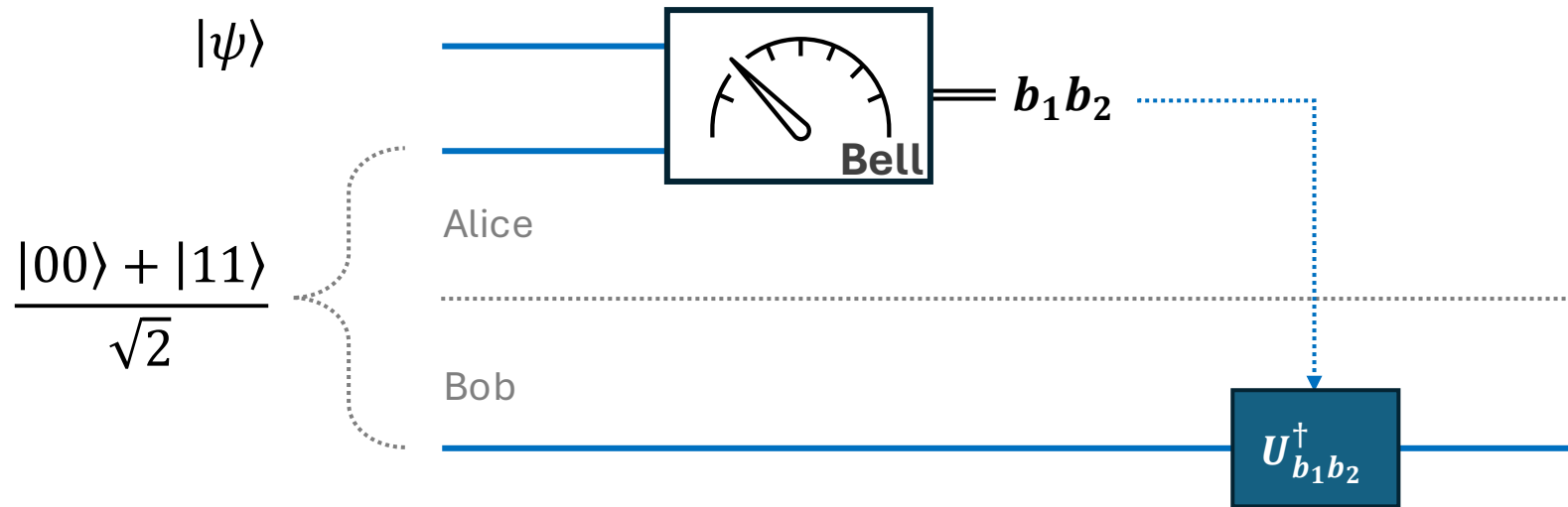


- $\rho' = |b\rangle\langle b|$

- If Bob sends  $b$  to Alice, then  $\rho' = |b\rangle\langle b|$ .
- If not,  $\rho' = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$  in Alice's view. (No information about  $b$  at all!)

# Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state  $|\psi\rangle$ .

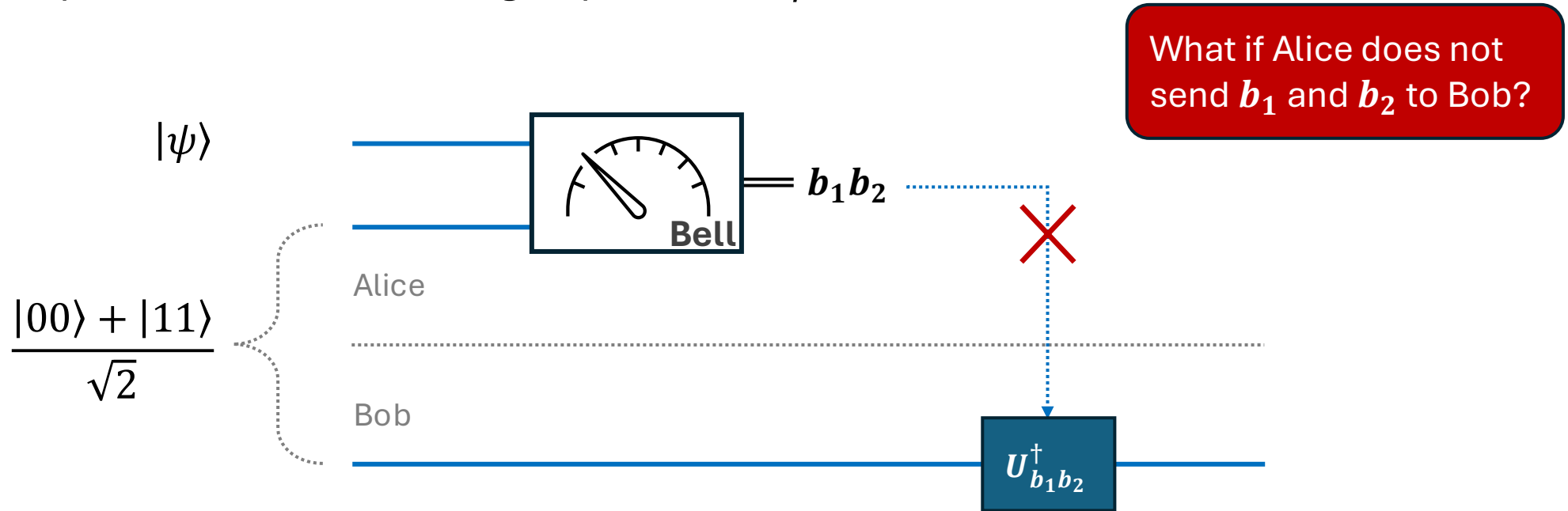


**Total states:**

$$|\phi_0\rangle = \frac{1}{2} \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2} |\psi\rangle \quad |\phi_1\rangle = |\beta_{b_1 b_2}\rangle U_{b_1 b_2} |\psi\rangle \quad |\phi_2\rangle = |\beta_{b_1 b_2}\rangle \otimes |\psi\rangle$$

# Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state  $|\psi\rangle$ .



What if Alice does not send  $b_1$  and  $b_2$  to Bob?

**Total states:**

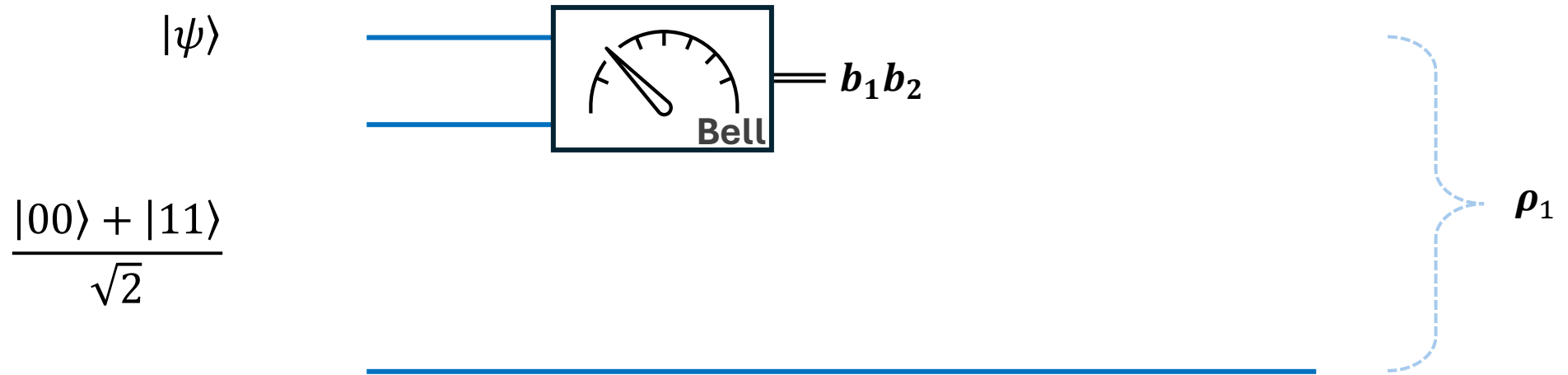
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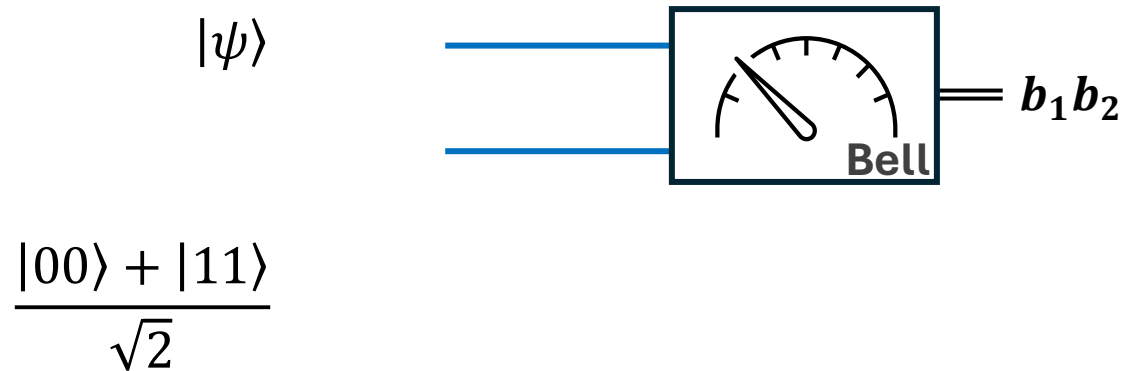


**Total states:**

$$|\phi_0\rangle = \frac{1}{2} \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2} |\psi\rangle \quad \rho_1 = ?$$

# Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state  $|\psi\rangle$ .



If  $b_1 b_2 = 00$  (with probability  $\frac{1}{4}$ ),  
then the total state is

$$|\beta_{00}\rangle U_{00}^\dagger |\psi\rangle,$$

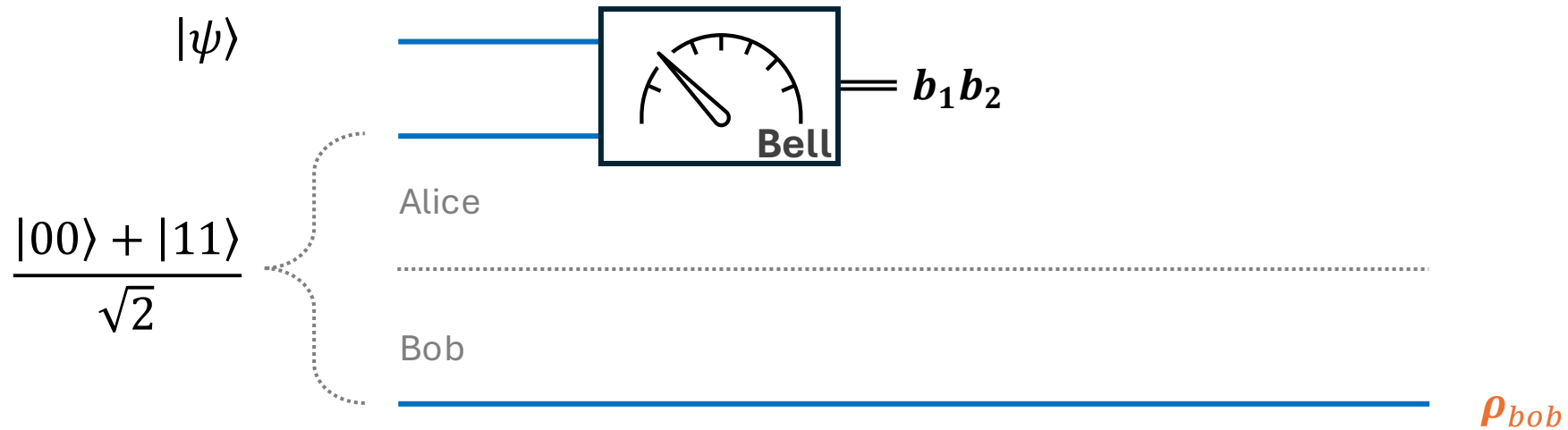
and similarly...

**Total states:**

$$\begin{aligned}
 |\phi_0\rangle &= \frac{1}{2} \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2} |\psi\rangle \\
 \rho_1 \text{ (the density operator on Bob's view after Alice's measurement)} &= \frac{1}{4} \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2} |\psi\rangle \langle \beta_{b_1 b_2}| U_{b_1 b_2}^\dagger \langle \psi|
 \end{aligned}$$

# Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state  $|\psi\rangle$ .

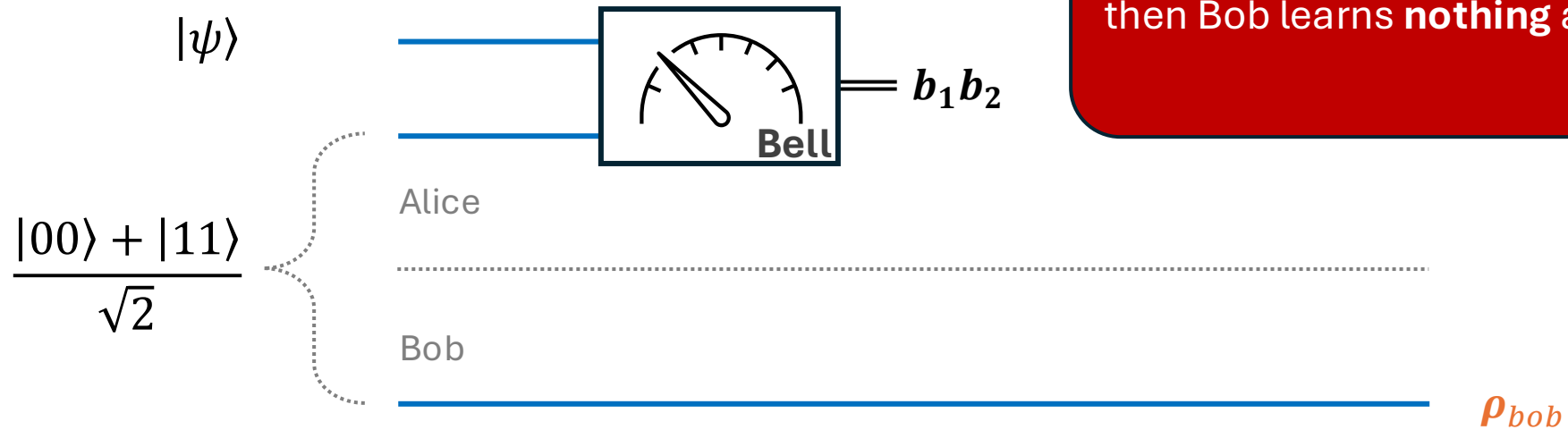


$$\rho_{bob} = \text{tr}_{alice}(\rho_1) = \text{tr}_{alice} \left( \frac{1}{4} \sum_{b_1, b_2 \in \{0,1\}} |\beta_{b_1 b_2}\rangle U_{b_1 b_2} |\psi\rangle \langle \beta_{b_1 b_2}| U_{b_1 b_2}^\dagger \langle \psi| \right)$$

$$= I/2$$

# Another look at Quantum Teleportation

- Quantum teleportation: Transmit a single-qubit state  $|\psi\rangle$ .



If Alice does not send  $b_1 b_2$  to Bob, then Bob learns **nothing** about  $|\psi\rangle$

$$\rho_{bob} = I/2 \text{ (independent of } |\psi\rangle\text{)!}$$

# Reference

- **[NC00]:** Sections 2.4, 8.3.1, and 9.2.1
- **[KLM07]:** Section 3.5
  
- **Purification:** [NC00, Section 2.5] and [KLM07, Section 3.5.2]

# Next Topics

- Quantum key distribution
- Quantum money