

Quantum Computing

- Week 2 (April 21-22, 2026)
- Today:
 - Quantum state, qubit, and their linear algebra formulation

Qubit

- A **qubit** describes the quantum state of a quantum system
- Abstracted as a mathematical object (i.e., ignore their physical meanings...)
- Two “basic” states $|0\rangle, |1\rangle$
 - Dirac (Bra-ket) notations
 - In some research papers, $| \rangle$ is also called a *quantum register*
- We describe the **superposition** state of the system using the qubit:

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- The numbers α and β are **complex number** and $|\alpha|^2 + |\beta|^2 = 1$

Qubit

- We describe the state of a system using the **single** qubit:

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- The numbers α and β are **complex number** and $|\alpha|^2 + |\beta|^2 = 1$

Qubit

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Superposition (for single qubit, informal): $|\phi\rangle$ cannot be written as either $|0\rangle$ or $|1\rangle$

Qubit

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A quick recap of complex numbers \mathbb{C} :

- A complex number $\alpha \in \mathbb{C}$ can be written as $\alpha = a + bi$, where a, b are real numbers, and $i = \sqrt{-1}$
- If $\alpha \in \mathbb{C}$ and $\alpha = a + bi$, then we write its **conjugate** as $\alpha^* = a - bi$
- We write α 's **norm** as $|\alpha| = \sqrt{a^2 + b^2}$. We always have $|\alpha| = |\alpha^*| = \sqrt{\alpha\alpha^*}$
- If $|\alpha| = 1$, then α can also be written as $\alpha = \cos \theta + i \sin \theta$ for some θ .
- By Euler's formula, $\alpha = \cos x + i \sin x = e^{ix}$, and $|e^{ix}| = 1$

Qubit

- We describe the state of a system using the **single** qubit:

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- The numbers α and β are **complex number** and $|\alpha|^2 + |\beta|^2 = 1$

- **Examples:**

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\cos \theta |0\rangle + e^{i\psi} \sin \theta |1\rangle$$

Qubit as a unit vector

- We describe the state of a system using the **single** qubit:

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- The numbers α and β are **complex number** and $|\alpha|^2 + |\beta|^2 = 1$

- **Relation between $|0\rangle$ and $|1\rangle$:**

- They should be “**easy to distinguish**”
- Linear algebra representation:

$$|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Qubit as a unit vector

- Some linear algebra:
 - Focus on vector spaces over \mathbb{C}
 - Linear (in)dependence, basis, orthonormal basis, transpose, adjoint, ...
- $|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\langle 0| := [1^* \ 0^*](= [1 \ 0])$, or more generally, if $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, then $\langle\psi| = [\alpha^* \ \beta^*]$
- We call $|\psi\rangle$ a “**ket**” and $\langle\psi|$ a “**bra**”
- Inner product using Dirac (Bra-ket) notations: $\langle\phi|\psi\rangle$
- Easy to see $\langle 0|1\rangle = \langle 1|0\rangle = 0$ and $\langle 0|0\rangle = 1 = \langle 1|1\rangle$

Qubit as a unit vector

- We describe the state of a system using the **single** qubit:
 - The numbers α and β are **complex numbers**

$$\begin{aligned} |\phi\rangle &= \alpha|0\rangle + \beta|1\rangle \\ &= \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2 \end{aligned}$$

- A single qubit is a **unit vector over \mathbb{C}^2**

$$\| |\phi\rangle \| = \sqrt{\langle \phi | \phi \rangle} = \sqrt{|\alpha|^2 + |\beta|^2} = 1$$

- Change basis:

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of \mathbb{C}^2 (known as **computational basis**)

$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is also a basis of \mathbb{C}^2

Qubit in Different Bases

- Single qubit: $|\phi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2, \|\phi\rangle\| = 1$
- Change basis: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of \mathbb{C}^2 (known as the **computational basis**)
 $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is also a basis of \mathbb{C}^2 .
- Let $|+\rangle := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $|-\rangle := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, then:

$$|\phi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\alpha+\beta}{\sqrt{2}} |+\rangle + \frac{\alpha-\beta}{\sqrt{2}} |-\rangle$$

Qubit in Different Bases

- Single qubit: $|\phi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2, \|\phi\rangle\| = 1$

- Described by different bases:

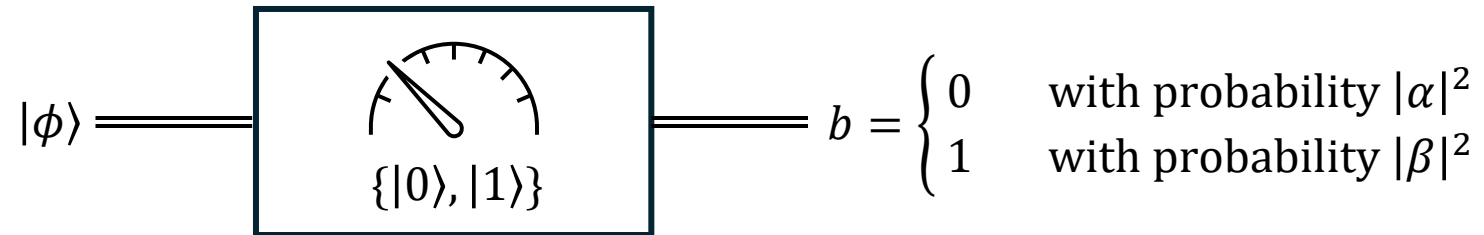
$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\phi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$$

- What do they mean? **Depends on measurement** (will be introduced later)

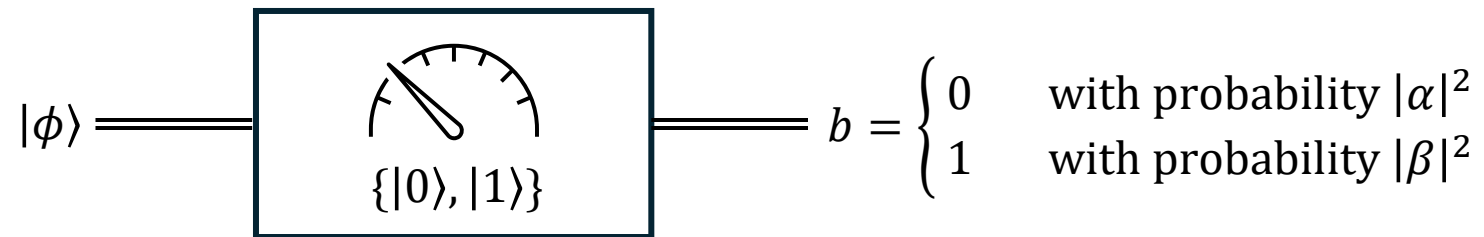
Single qubit measurement

- Single qubit: $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2$
- If we measure $|\phi\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$:



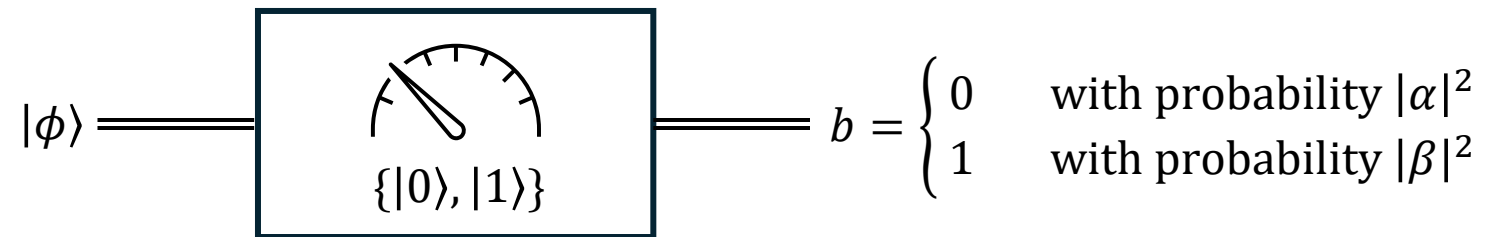
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Single qubit measurement

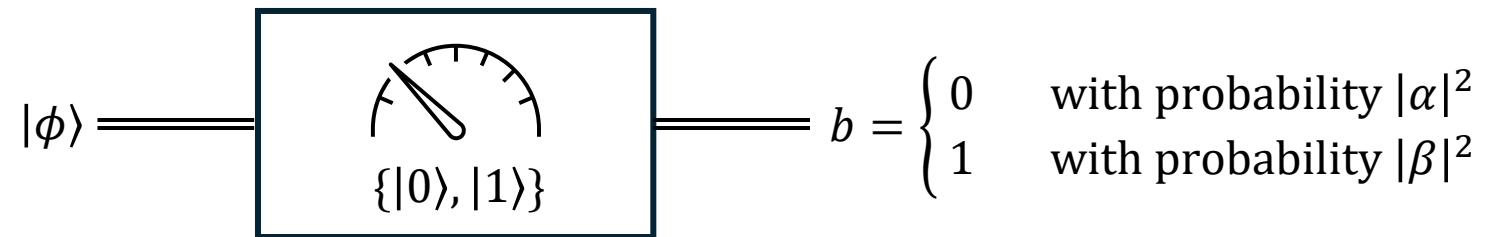
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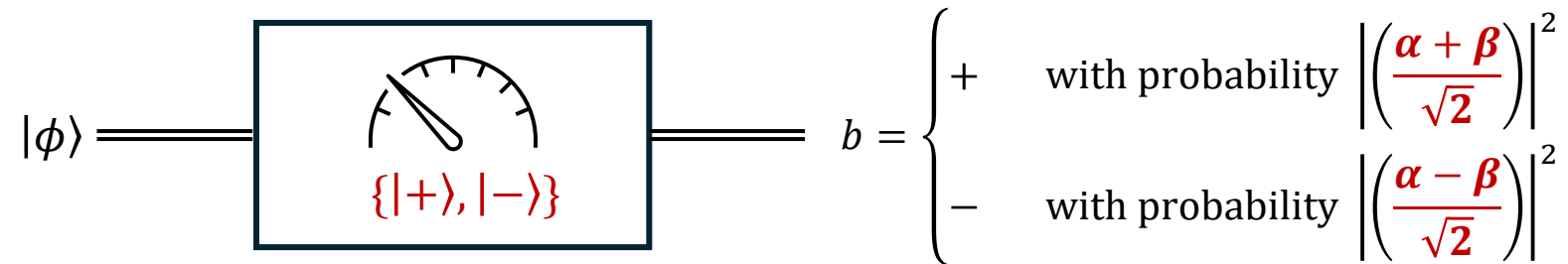
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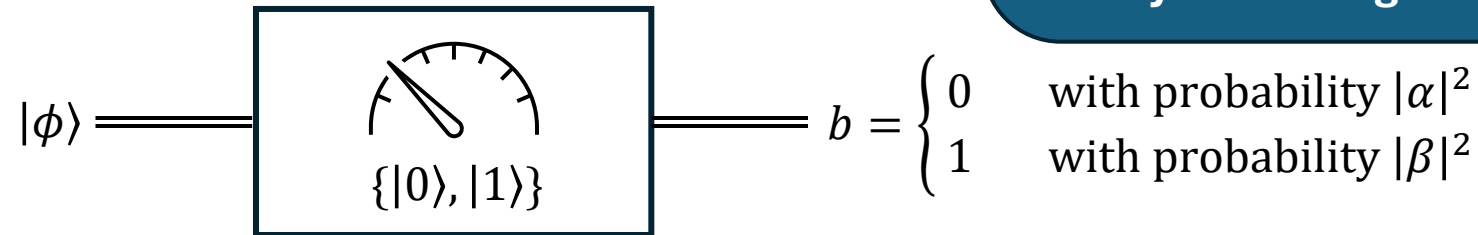


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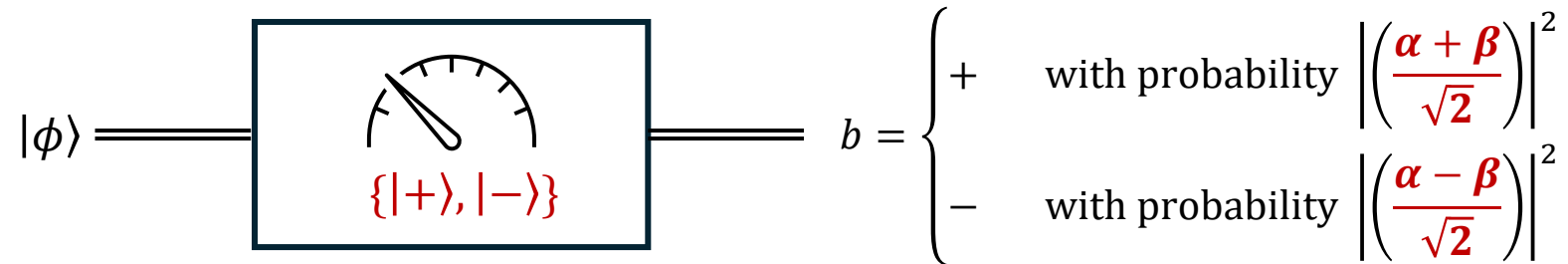


Single qubit measurement

- Single qubit: $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$
- If we measure $|\phi\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$:



- If we measure $|\phi\rangle$ in the basis $\{|+\rangle, |-\rangle\}$:



- Measurement outcomes depend on the chosen basis.
- The same quantum state can give different statistics under different measurement bases.
- A basis corresponds to a particular way of reading out the qubit.

Single qubit measurement

- Single qubit: $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

Notes:

1. We may also call α and β as amplitudes
2. Why complex numbers? Quantum states need amplitudes with phases.

Single qubit measurement

- Single qubit: $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

Wrong: The qubit is $|0\rangle$ with probability $|\alpha|^2$ and is $|1\rangle$ with probability $|\beta|^2$

The qubit is in a superposition before measurement: in both $|0\rangle$ and $|1\rangle$ at once

Single qubit measurement

- Single qubit: $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

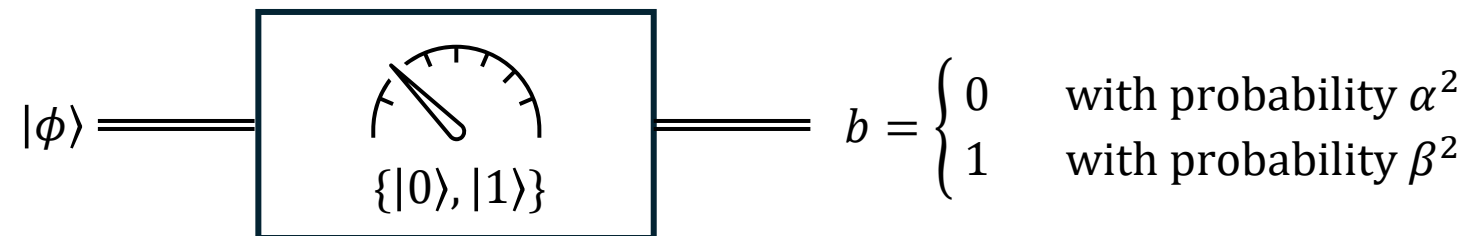
Can we estimate α and β by measuring $|\phi\rangle$ many times?

Single qubit measurement

- Single qubit: $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

Can we estimate α and β by measuring $|\phi\rangle$ many times?

No. Because of collapse and no-cloning...



$|\phi\rangle$ becomes $|b\rangle$ after measurement...

Inner/Outer Product

- Let $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ be a qubit
- Inner product (to see adjoint and linearity):

$$\langle\phi|\phi\rangle = \langle\phi| \cdot |\phi\rangle = (\alpha^*\langle 0| + \beta^*\langle 1|) \cdot (\alpha|0\rangle + \beta|1\rangle) = \dots = 1$$

- Outer product: $|\phi\rangle\langle\phi|$

$$|\phi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \langle\phi| = [\alpha^* \ \beta^*], |\phi\rangle\langle\phi| = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot [\alpha^* \ \beta^*] = (\text{a } 2 \times 2 \text{ matrix})$$

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What does $|\phi\rangle\langle\phi|$ represents? A **projector** that project a vector onto the “line” (one-dimension linear space) spanned by $|\phi\rangle$.

Tensor Product

- Let \mathbf{A} ($n_1 \times m_1$) and \mathbf{B} ($n_2 \times m_2$) be two arbitrary complex matrices, where

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,m_1} \\ \vdots & \ddots & \vdots \\ a_{n_1,1} & \cdots & a_{n_1,m_1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{1,1} & \cdots & b_{1,m_2} \\ \vdots & \ddots & \vdots \\ b_{n_2,1} & \cdots & b_{n_2,m_2} \end{bmatrix}$$

- Then the **tensor product** of \mathbf{A} and \mathbf{B} , denoted as $\mathbf{A} \otimes \mathbf{B}$, is defined by

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{1,1}\mathbf{B} & \cdots & a_{1,m_1}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{n_1,1}\mathbf{B} & \cdots & a_{n_1,m_1}\mathbf{B} \end{bmatrix}, \text{ which is a } n_1 n_2 \times m_1 m_2 \text{ matrix}$$

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- One can define **tensor product for vectors** in a natural way.
- We use tensor product to define **multiple qubits**

Multiple Qubits

- In the classical world, an n -bit string has 2^n possibilities (i.e., 2^n basic states)
- We define multiple qubits (in the **computational basis**) by an analogous way.

Multiple Qubits

- Multiple qubits in the **computational basis**.
- 2^n basic states: $|00 \cdots 00\rangle, |00 \cdots 01\rangle, |00 \cdots 10\rangle, |00 \cdots 11\rangle, \dots, |11 \cdots 11\rangle$, where

$$|b_{n-1}b_{n-2} \cdots b_1b_0\rangle := |b_{n-1}\rangle \otimes |b_{n-2}\rangle \otimes \cdots \otimes |b_1\rangle \otimes |b_0\rangle$$

- More compact representation:

$$|0\rangle, |1\rangle, |2\rangle, |3\rangle, \dots, |2^n - 1\rangle$$

- An **n -qubit states**: A **superposition** of the 2^n basic states (also a **unit vector over \mathbb{C}^{2^n}**)

$$|\phi\rangle = \sum_{i=0}^{2^n-1} \alpha_i |i\rangle,$$

$$\text{where } \alpha_i \in \mathbb{C} \text{ and } \sum_{i=0}^{2^n-1} |\alpha_i|^2 = 1$$

Multiple Qubits

- Multiple qubits in an arbitrary orthonormal basis: $|\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_{N-1}\rangle$
- A more general representation:

$$|\phi\rangle = \sum_{i=0}^{N-1} \alpha_i |\phi_i\rangle,$$

where $\alpha_i \in \mathbb{C}$ and $\sum_{i=0}^{N-1} |\alpha_i|^2 = 1$

References

- **[NC00]** *Quantum Computation and Quantum Information*. Michael Nielsen and Isaac Chuang
 - Section 1.2 (**Bloch sphere representation** of a qubit)
 - Sections 2.1.1 – 2.1.3
- **[KLM07]** *An Introduction to Quantum Computing*. Phillip Kaye, Raymond Laflamme, Michele Mosca
 - Sections 2.1, 2.2, and 2.6
- **[RP11]** *Quantum Computing: A Gentle Introduction*. Eleanor Rieffel and Wolfgang Polak
 - Sections 2.1-2.2, 3.1
- Professor Mark Zhandry's [lecture note](#).
- Professor Henry Yuen's [lecture note](#).