Quantum Computing

- Week 4 (May 5-6, 2026)

- Today:
  - Postulates of Quantum Computing

Postulates of Quantum Computing

"Don't be surprised if the motivation for the postulates is not always clear; even to experts the basic postulates of quantum mechanics appear surprising..." from [NC00]

Postulates of Quantum Computing

- First postulate: State space

> **Postulate 1**: Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space* of the system. The system is completely described by its *state vector*, which is a unit vector in the system's state space.
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> from [NC00]

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- Keywords:
  - **Isolated system**
  - **Hilbert space**: Complex inner product linear space (e.g., $\mathbb{C}^{2^n}$)
  - The state of a system is completely described by a **state vector**
  - A state vector is a **unit vector of** the Hilbert space

Example:
$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

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  - **Isolated system: (Informally,) Not entangled with other systems…**
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Postulates of Quantum Computing

- Second postulate: Evolution

> **Postulate 2**: The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2,
>
> $$|\psi'\rangle = U|\psi\rangle \, .$$
>
> from [NC00]

- Keywords:
  - **Closed system**
  - **Unitary transformation**

Postulates of Quantum Computing

- Second postulate (using Schrodinger's equation): Evolution

Postulate 2′: The time evolution of the state of a closed quantum system is described by the *Schrödinger equation*,

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

In this equation, $\hbar$ is a physical constant known as *Planck's constant* whose value must be experimentally determined. The exact value is not important to us. In practice, it is common to absorb the factor $\hbar$ into H, effectively setting $\hbar = 1$. H is a fixed Hermitian operator known as the *Hamiltonian* of the closed system.

from [NC00]

Postulates of Quantum Computing

- Fourth postulate: Composite system:

> **Postulate 4**: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

from [NC00]

Postulates of Quantum Computing

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- Keywords:
  - (Suppose each component system is **isolated**)
  - **Tensor product**
  - State space of the whole system: **Tensor product of the Hilbert spaces** of each component system
  - State vector of the whole system: **Tensor product of the state vector** of each component system

Postulates of Quantum Computing

- Third postulate: Quantum measurement

- Do it on the board

- Measurement **in the computational basis**
- **Partial measurement**

Postulates of Quantum Computing

- Third postulate: Quantum measurement

- Measurement in the computational basis
- Partial measurement

- **Collapse: The state after measurement**

$$|\phi\rangle \longrightarrow \frac{M_m|\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}}$$

Quantum Measurement

- Let $\{M_m\}_m$ be a set of matrices describing some quantum measurement

- Let $|\phi\rangle$ be a quantum state, perform the same measurement $\{M_m\}_m$ on $|\phi\rangle$ **twice**

$$|\phi\rangle \longrightarrow \frac{M_m|\phi\rangle}{\sqrt{\langle\phi|M_m^\dagger M_m|\phi\rangle}} \longrightarrow ?$$

(with probability
$p(m) = \langle\phi|M_m^\dagger M_m|\phi\rangle$)

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We need more restrictions to achieve **“Stability”**

Projective Measurement

- Projective measurements: A special class of measurements

Projective Measurement

- Some Linear Algebra – **Projector and Eigenspace**:
  - A matrix $\boldsymbol{P}$ is a projector (or projection operator) if $\boldsymbol{P^2 = P}$
  - For any vector $\boldsymbol{x}$, $\boldsymbol{P^n x = P^{n-1} x = \cdots = Px}$

  - **Eigenvalues** and **Eigenvectors**: $\boldsymbol{Ax = \lambda x}$
  - A matrix (linear operator) may have **multiple eigenvalues**
  - Each eigenvalue may have **multiple linearly independent eigenvectors**

  - Let $\{\boldsymbol{x_1, \ldots, x_m}\}$ denote a *maximal set* of linearly independent eigenvectors of λ (i.e., $\boldsymbol{Ax_i = \lambda x_i}$)
  - We say $\{\boldsymbol{x_1, \ldots, x_m}\}$ **span the eigenspace** of $\boldsymbol{A}$ with eigenvalue λ

  - Given such $\{\boldsymbol{x_1, \ldots, x_m}\}$, the **Gram-Schmidt process** gives us an **orthogonal basis of the eigenspace**
  - Given an orthogonal basis, we can easily compute the **projector onto this eigenspace**

Projective Measurement

- Some Linear Algebra – **Spectral decomposition (simplified)**:

  Any Hermitian operator $\boldsymbol{M}$ (i.e., $\boldsymbol{M} = \boldsymbol{M}^{\dagger}$) can be written as:

$$\boldsymbol{M} = \sum_{\lambda} \lambda \boldsymbol{P}_{\lambda}$$

  - λ represents an eigenvalue of $\boldsymbol{M}$
  - $\boldsymbol{P}_{\lambda}$ represents the projector onto the λ eigenspace
  - $\boldsymbol{P}_{\lambda}$ itself is also Hermitian, i.e., $\boldsymbol{P}_{\lambda} = \boldsymbol{P}_{\lambda}^{\dagger}$

  - Examples (show on the board)…

Projective Measurement

- Projective measurements: A special class of measurements
- (Do it on the board)

- Keywords
  - **Observable** $\boldsymbol{M} = \sum_m m\boldsymbol{P}_m$ is a *Hermitian matrix*
    - m represents an eigenvalue of $\boldsymbol{M}$**,** and it is also used to **label a measurement outcome**
    - $\boldsymbol{P}_m$ represents the projector onto the m eigenspace
    - $\boldsymbol{P}_m$ is also Hermitian
  - Measurement outcome **correspond to the eigenvalues**
    - e.g., $p(m) = \langle\phi|P_m|\phi\rangle$
  - The state after measurement: $|\phi\rangle \rightarrow \frac{P_m|\phi\rangle}{\sqrt{p(m)}}$

Projective Measurement

- Projective measurements: A special class of measurements

- Relation to Postulate 3:
  - $\boldsymbol{M} = \sum_m m\boldsymbol{P}_m$, but $\sum_m \boldsymbol{P}_m = \boldsymbol{I}$, so the completeness condition holds
  - Note: An eigenvalue of an observable just **represents a possible outcome (i.e., a label), but not the probability or physical meaning by itself**

- Examples: $|0\rangle\langle 0| = 1 \cdot |0\rangle\langle 0| + 0 \cdot |1\rangle\langle 1|$ and $|1\rangle\langle 1| = 0 \cdot |0\rangle\langle 0| + 1 \cdot |1\rangle\langle 1|$
  - Both can be used to represent measurement in the computational basis

Projective Measurement

- Let $\boldsymbol{M} = \sum_m m\boldsymbol{P}_m$ be an observable
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(with probability $p(m) = \langle\phi|P_m|\phi\rangle$)

(with probability 1)

Non-Projective Measurement

- Let $\{M_m\}_m$ be a set of matrices describing some quantum measurement

- General measurement (Postulate 3): $M_m^\dagger M_m$ is not necessarily a projector.
  - **Non-projective measurement:** $M_m^\dagger M_m$ is not a projector
  - Cannot guarantee that the same result will be reproduced if the same measurement is repeated

- Used in various quantum information-processing protocols
  - But will not be not covered in this course

The Deutsch-Jozsa Problem

- Constant-vs-balanced problem
- Let $f: \{0,1\}^n \rightarrow \{0,1\}$ be a bit function such that it is in either two cases:
  - f is a *constant* function: $\forall \boldsymbol{x} \in \{0,1\}^n, f(\boldsymbol{x})$ is always a constant (0 or 1)
  - f is a *balanced* function: $\sum_{\boldsymbol{x} \in \{0,1\}^n} f(\boldsymbol{x}) = \boldsymbol{2^{n-1}}$ (i.e., outputs 0 for half the inputs, and 1 for the other half)

- To decide whether f is constant or balanced, **how many times** must we evaluate f?

Classical Computer

Worst-case: 2^n

Probabilistic algorithm:

$l \ll 2^n$ times,

with a failure rate of $\frac{1}{2^l}$

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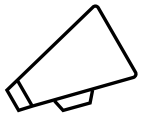
Can we do better?

References

- [NC00]: Sections 1.4.4, 2.2
- [KLM07]: Chapter 3, Sections 6.4.



No lectures next week (May 12-13)