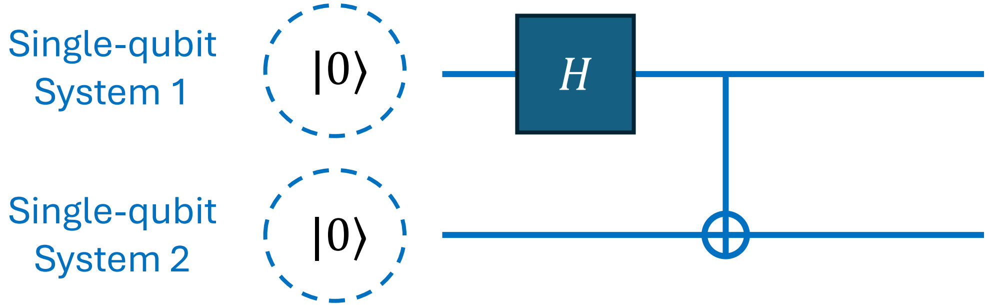
Quantum Computing

- Week 7 (May 26-27, 2026)

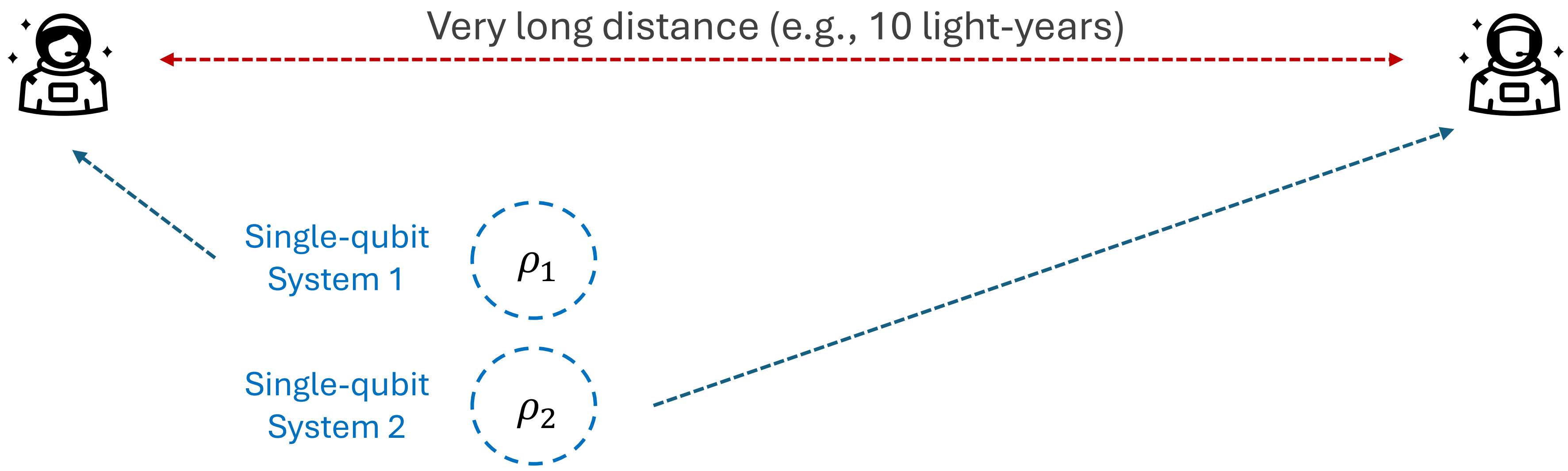
- Today:
  - Superdense coding
  - Quantum teleportation

Action at a Distance



$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Action at a Distance



Total state: $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$

Action at a Distance

Very long distance (e.g., 10 light-years)

Single-qubit
System 1

ρ_1

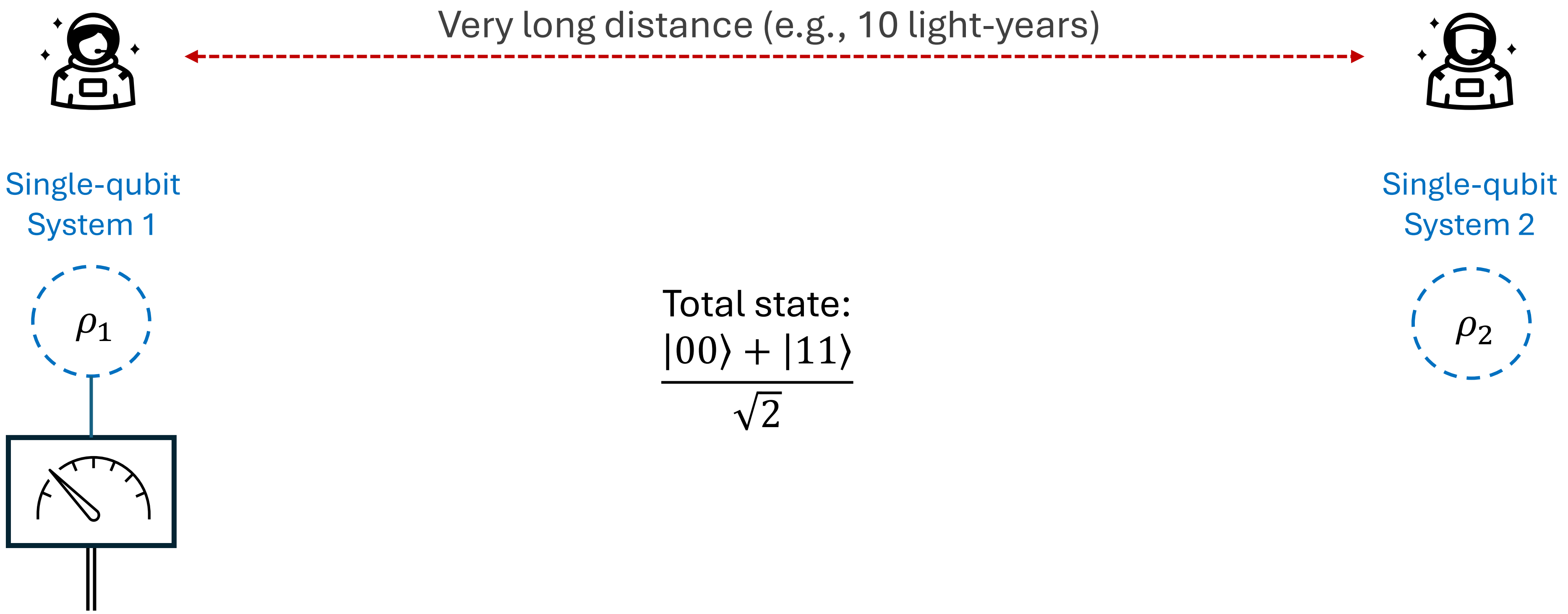
Single-qubit
System 2

ρ_2

Total state:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

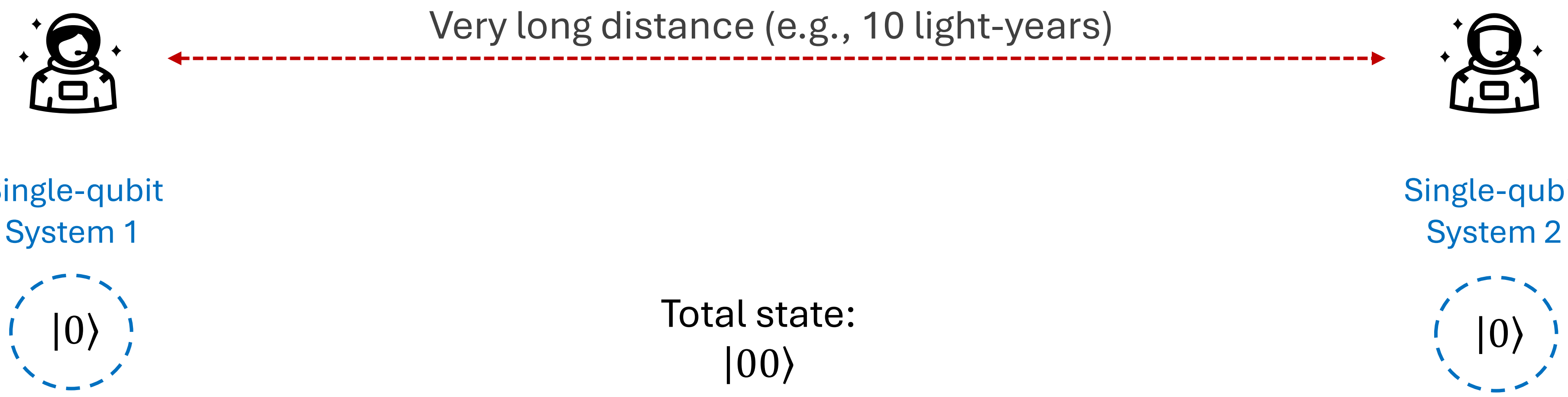
Action at a Distance



Total state:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Action at a Distance



- Application: Superdense coding

Pauli Matrices

- Pauli matrices:

$$X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{(Pauli-}\mathbf{X}\text{)} \qquad Y = \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{(Pauli-}\mathbf{Y}\text{)} \qquad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(Pauli-}\mathbf{Z}\text{)}$$

- Some facts:
  - Pauli-X is the qNOT gate (in the computational basis)
  - $\sigma_j^2 = I$ for $j = 1,2,3$

Pauli Matrices

- (Extended) Pauli matrices:

$$\boldsymbol{I} = \sigma_0 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{X} = \sigma_1 \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{Y} = \sigma_2 \coloneqq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \boldsymbol{Z} = \sigma_3 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(Pauli-**X**) (Pauli-**Y**) (Pauli-**Z**)

- Some facts:
  - Pauli-X is the qNOT gate (in the computational basis)
  - $\sigma_j^2 = I$ for $j = 0,1,2,3$

Superdense Coding

- Consider the four matrices

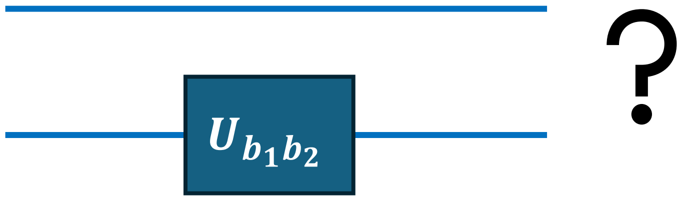
$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let’s define:

$$U_{b_1 b_2} := X^{b_2} Z^{b_1} \begin{cases} U_{00} = I \\ U_{01} = X \\ U_{10} = Z \\ U_{11} = XZ \end{cases}$$

- Small Exercise:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



Superdense Coding

- Consider the four matrices

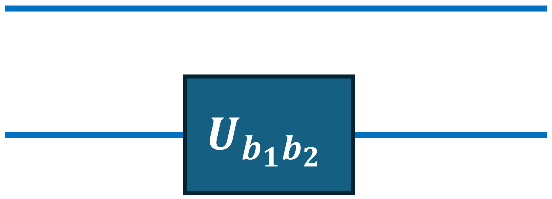
$$\boldsymbol{I} = \sigma_0 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \boldsymbol{X} = \sigma_1 \coloneqq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \boldsymbol{Z} = \sigma_3 \coloneqq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \boldsymbol{Z} \cdot \boldsymbol{X} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Let's define:

$$\boldsymbol{U}_{b_1 b_2} \coloneqq \boldsymbol{X}^{b_2} \boldsymbol{Z}^{b_1} \begin{cases} \boldsymbol{U}_{00} = \boldsymbol{I} \\ \boldsymbol{U}_{01} = \boldsymbol{X} \\ \boldsymbol{U}_{10} = \boldsymbol{Z} \\ \boldsymbol{U}_{11} = \boldsymbol{XZ} \end{cases}$$

- Small Exercise:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



$$|\beta_{b_1 b_2}\rangle = \frac{|0\, b_2\rangle + (-1)^{b_1} |1\, \overline{b_2}\rangle}{\sqrt{2}}$$

Superdense Coding

- Consider the four matrices

$$I = \sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Z = \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad Z \cdot X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

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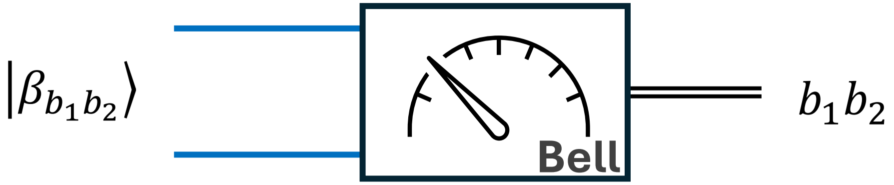
- Small Exercise:



Measurement
in the Bell basis

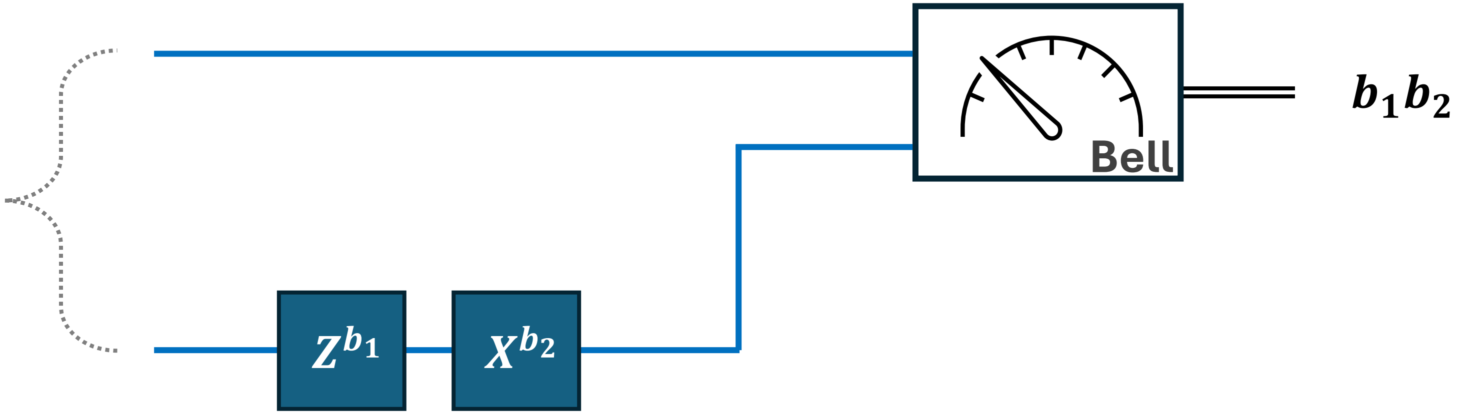
The Bell basis:

$$\begin{Bmatrix} M_{00}: |\beta_{00}\rangle\langle\beta_{00}|, \\ M_{01}: |\beta_{01}\rangle\langle\beta_{01}|, \\ M_{10}: |\beta_{10}\rangle\langle\beta_{10}|, \\ M_{11}: |\beta_{11}\rangle\langle\beta_{11}|, \end{Bmatrix}$$

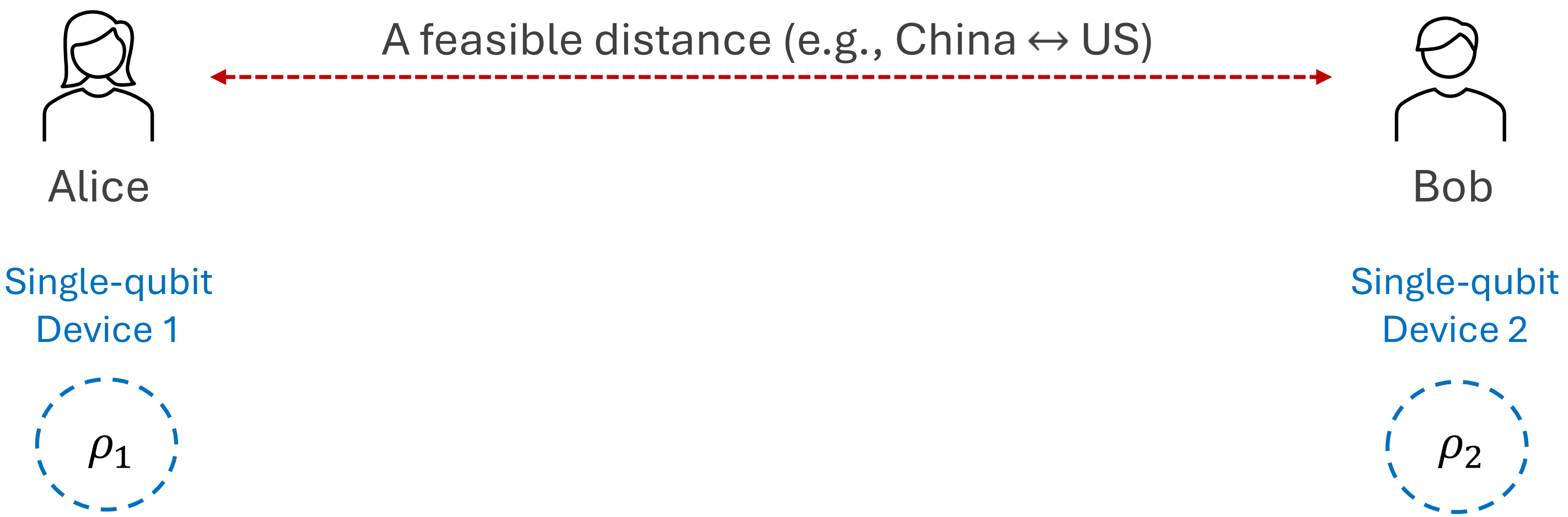
Superdense Coding

- A compact description of the experiment:

$$\beta_{00} := \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$



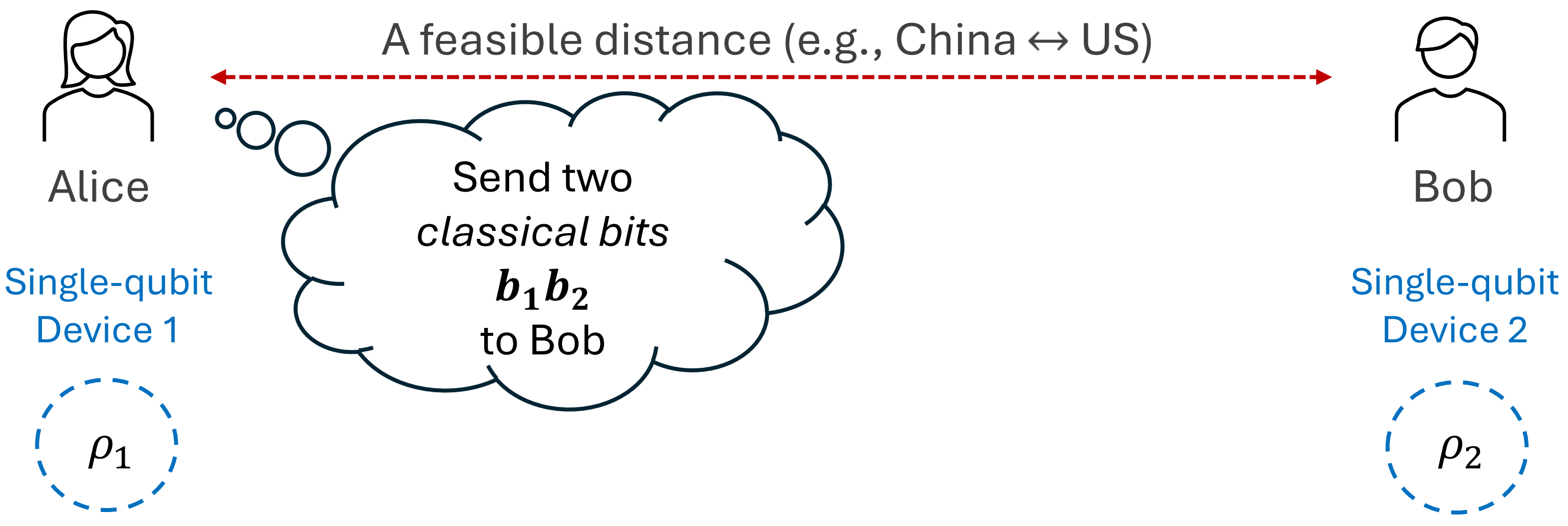
Superdense Coding



Total state:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Superdense Coding



Total state (before $\boldsymbol{U_{b_1 b_2}}$):

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Superdense Coding

Total state (before $\boldsymbol{U_{b_1 b_2}}$):

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Superdense Coding

A feasible distance (e.g., China ↔ US)

Alice

Bob

Single-qubit
Device 1

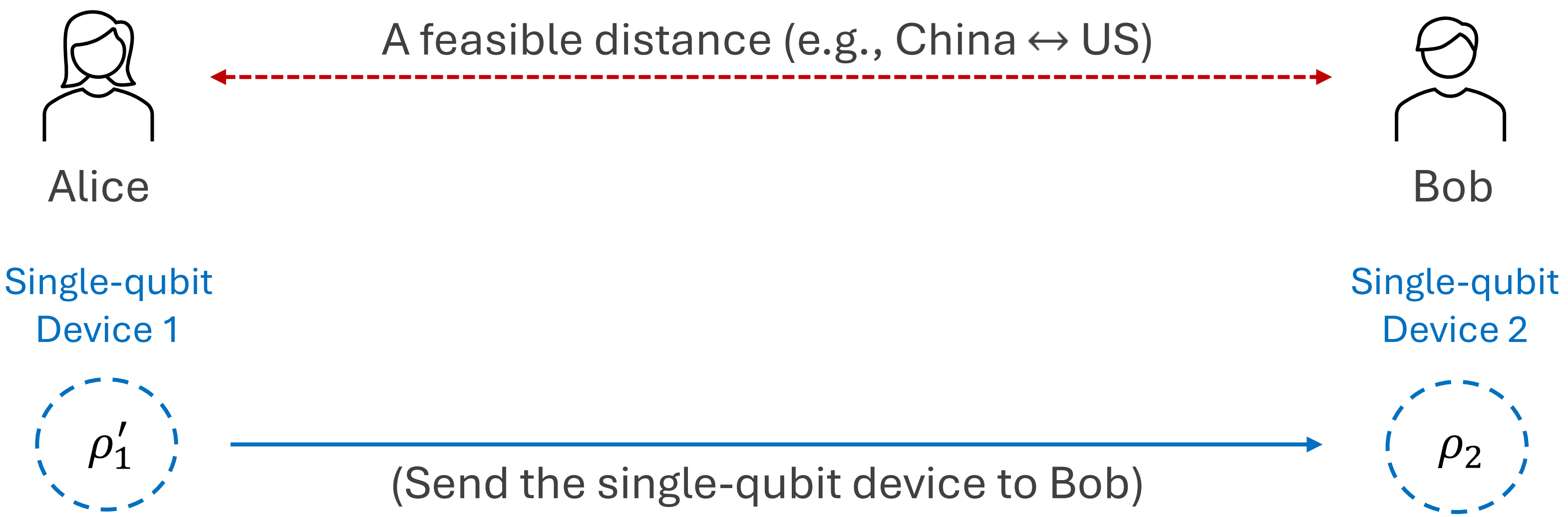
ρ_1'

Single-qubit
Device 2

ρ_2

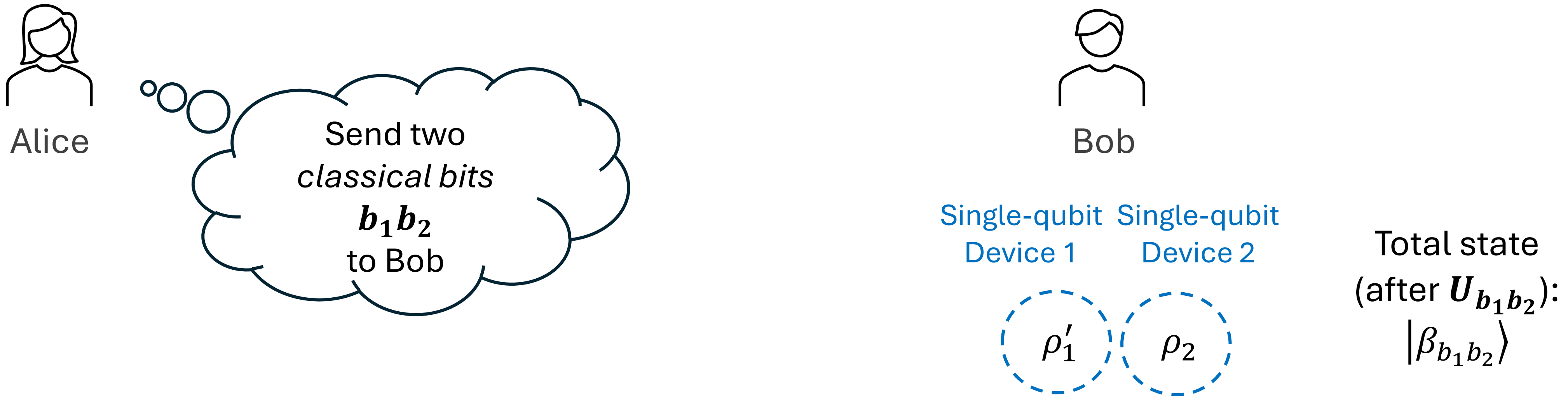
Total state
(after $\boldsymbol{U_{b_1 b_2}}$):
$|\beta_{b_1 b_2}\rangle$

Superdense Coding

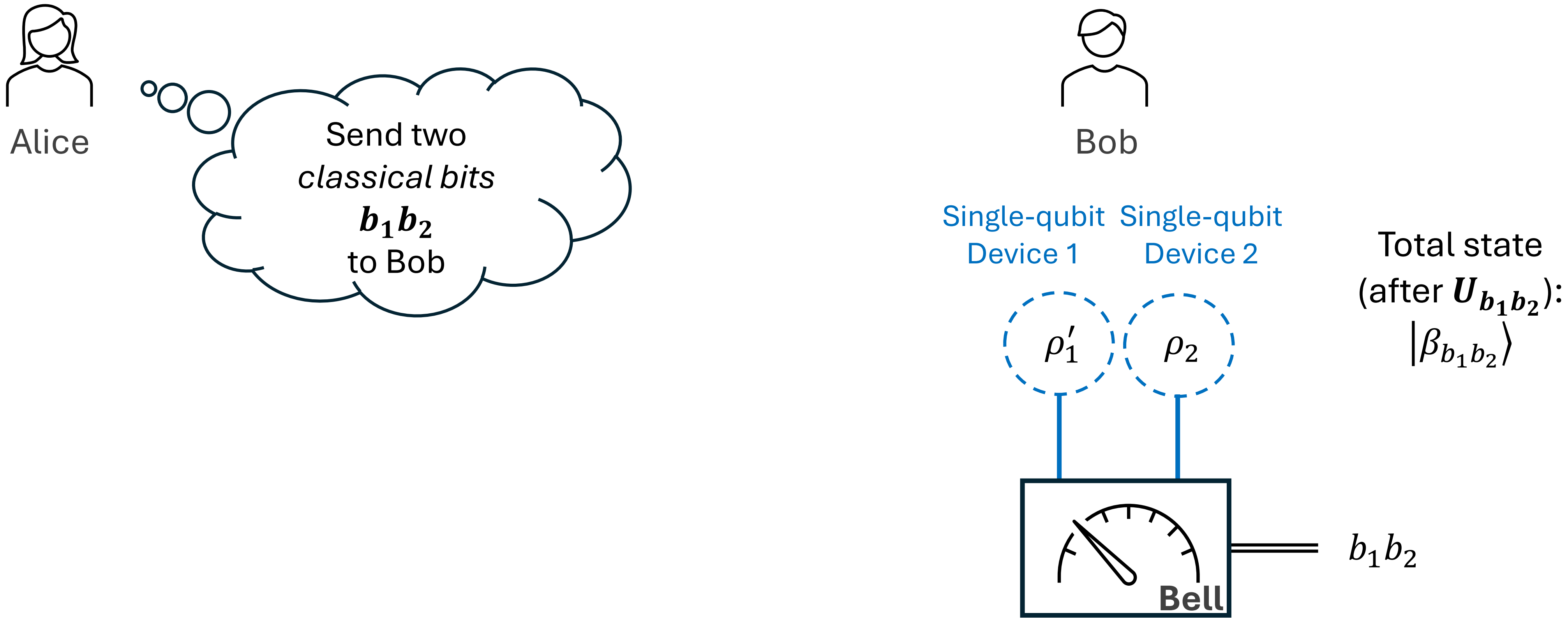


Total state
(after $\boldsymbol{U_{b_1 b_2}}$):
$|\beta_{b_1 b_2}\rangle$

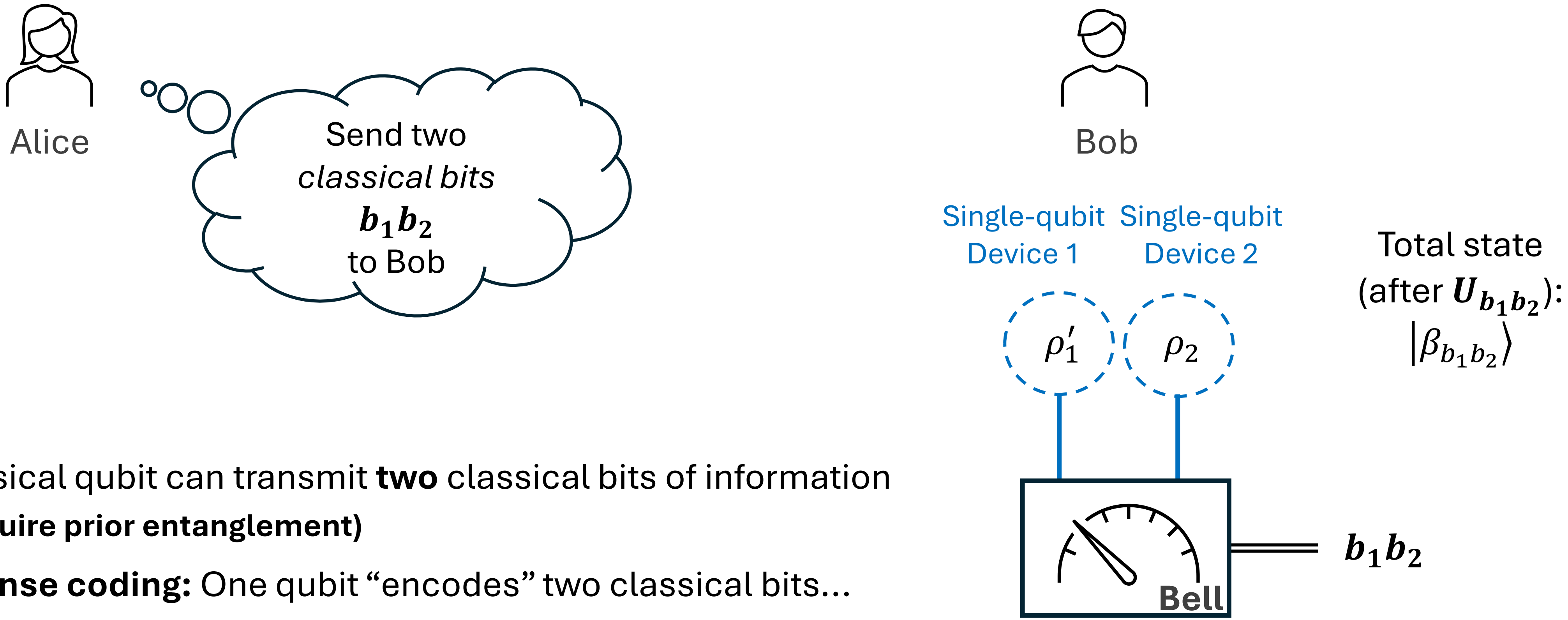
Superdense Coding



Superdense Coding



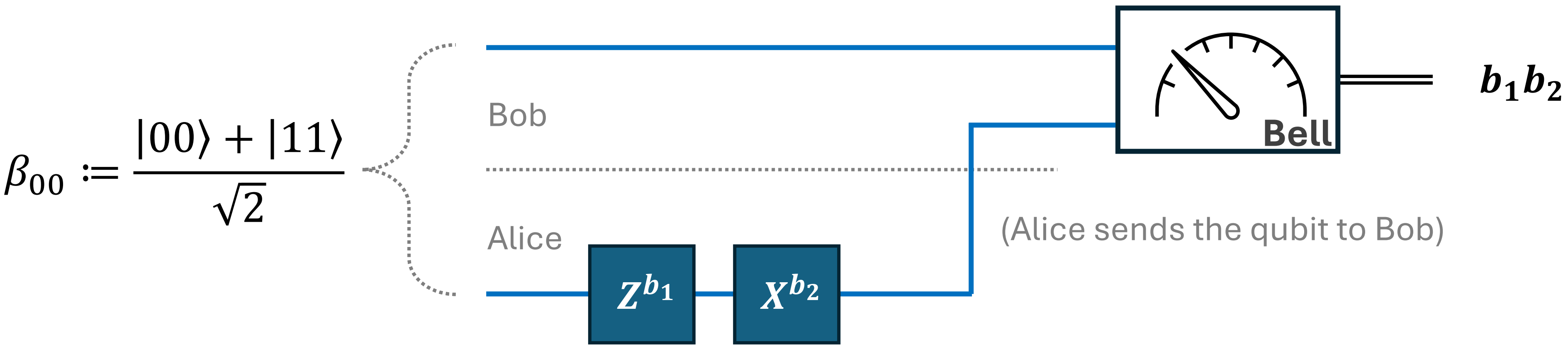
Superdense Coding



- **One** physical qubit can transmit **two** classical bits of information
  - **(Require prior entanglement)**
- **Superdense coding:** One qubit "encodes" two classical bits...

Superdense Coding

- A more compact description of the experiment:



- Quick question: How can we **transmit $2n$ classical bits using only n qubits?**

Quantum Teleportation

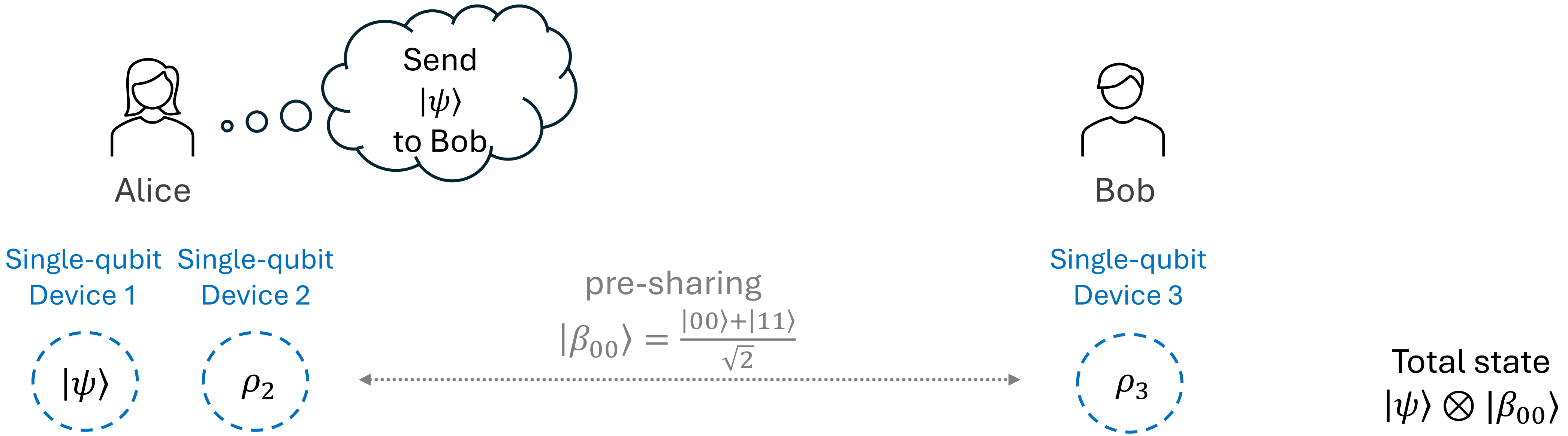
- Superdense coding: Transmit classical bits via qubits
- Quantum teleportation: **Transmit qubits via classical bits**

- A trivial approach (transportation): Physically deliver the device carrying $|\psi\rangle$ to Bob
- But here we consider **Quantum Teleportation**: Transmit via (quantum) channel

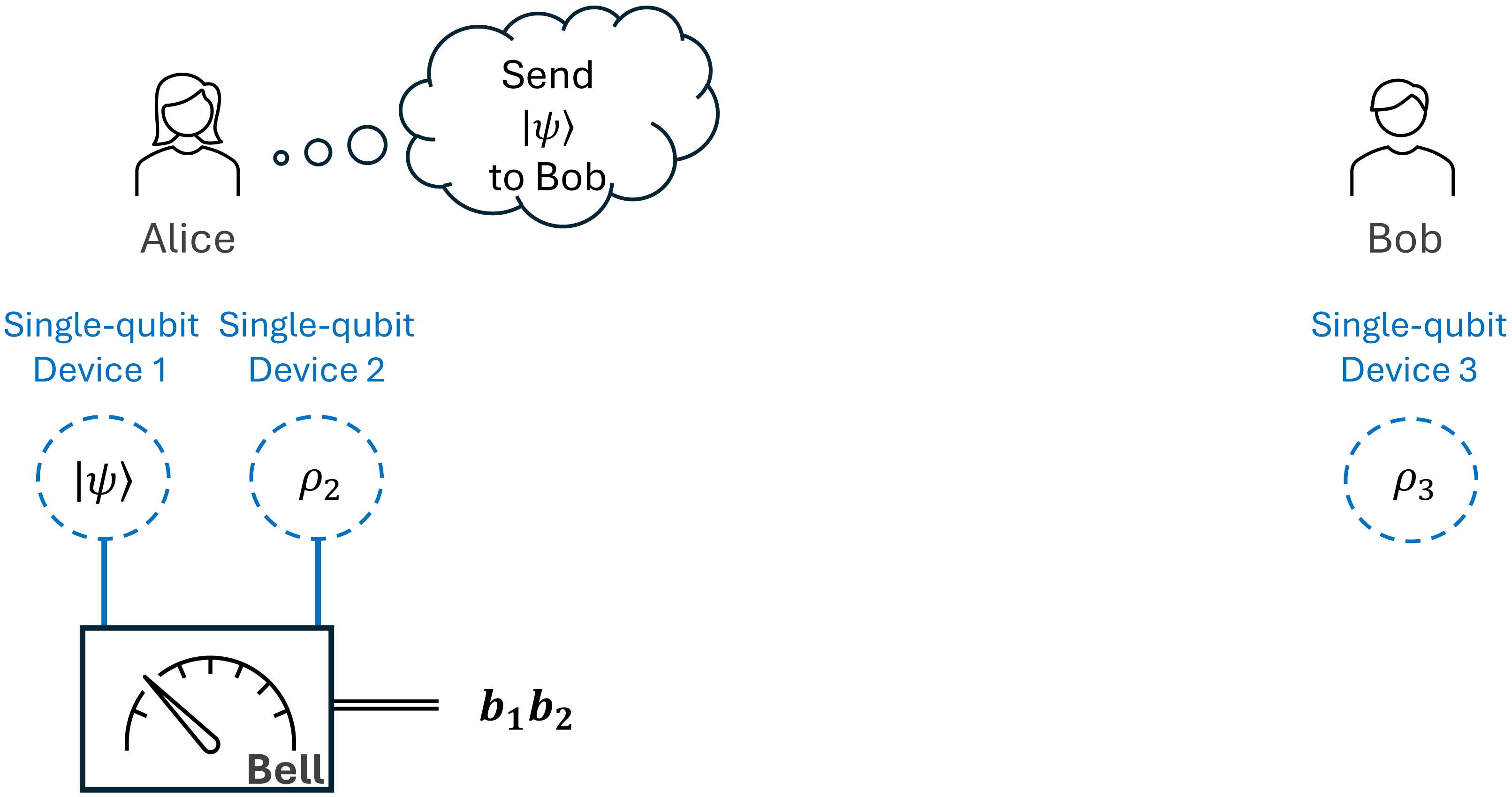
Quantum Teleportation

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Quantum Teleportation



Quantum Teleportation



Total state (before measurement)

$|\psi\rangle \otimes |\beta_{00}\rangle$

Quantum Teleportation

Alice

Send
$|\psi\rangle$
to Bob

Bob

Single-qubit Device 1

ρ'_1

Single-qubit Device 2

ρ'_2

Single-qubit Device 3

ρ'_3

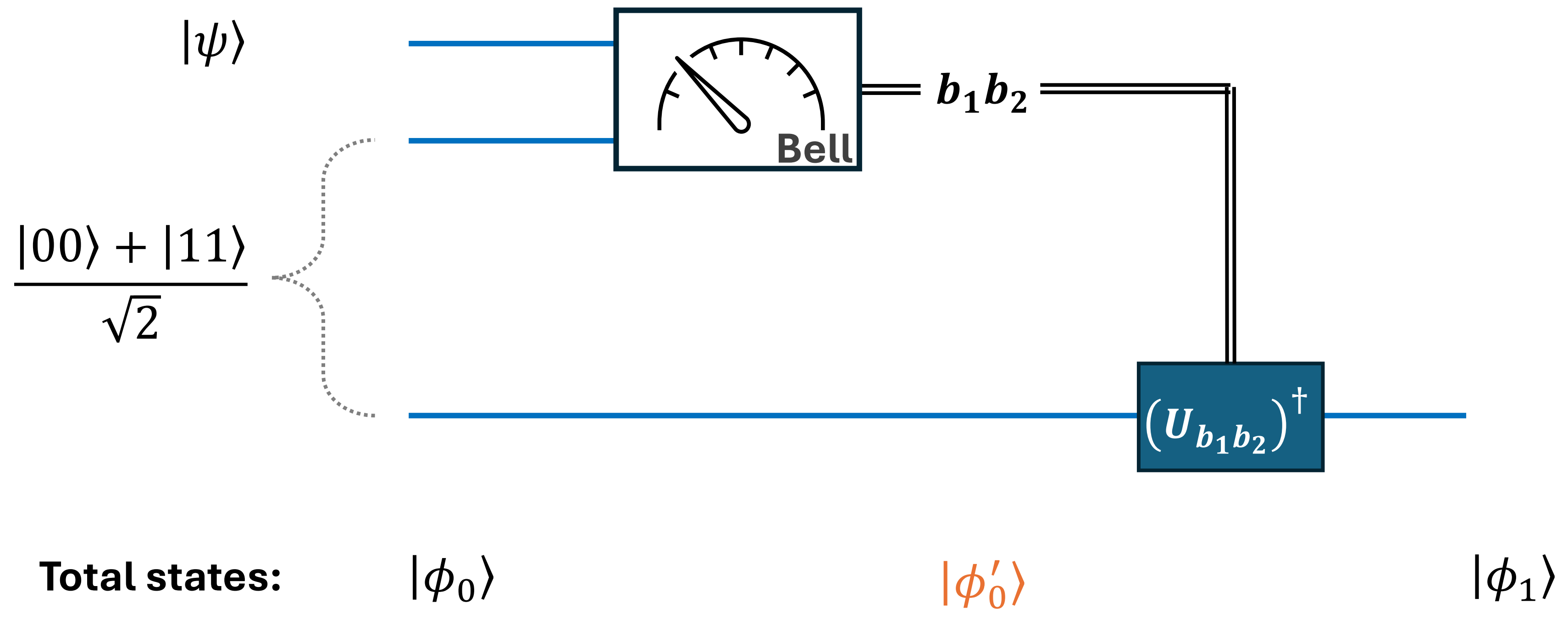
Total state (after measurement)

?

$\boldsymbol{b_1 b_2}$ → Send the classical bits via a classical channel

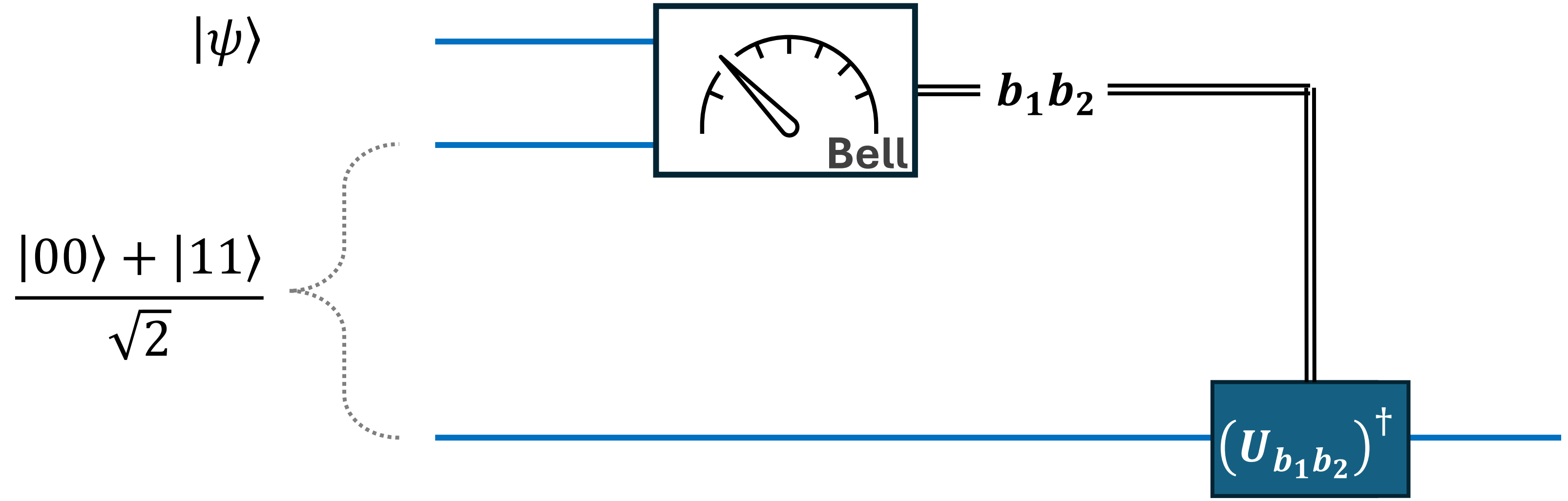
Quantum Teleportation

- **A more compact description (for analyzing states):**



Quantum Teleportation

- **A more compact description (for analyzing states):**

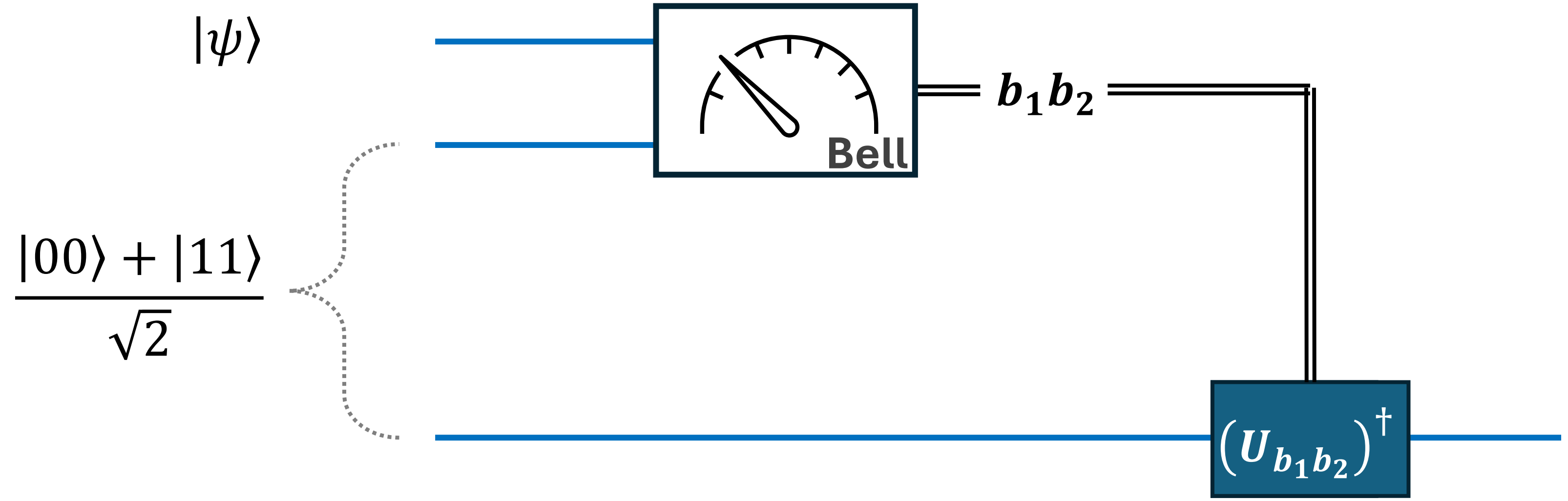


Total states: $|\phi_0\rangle$

Can we re-write the state of the first two systems using the Bell basis?

Quantum Teleportation

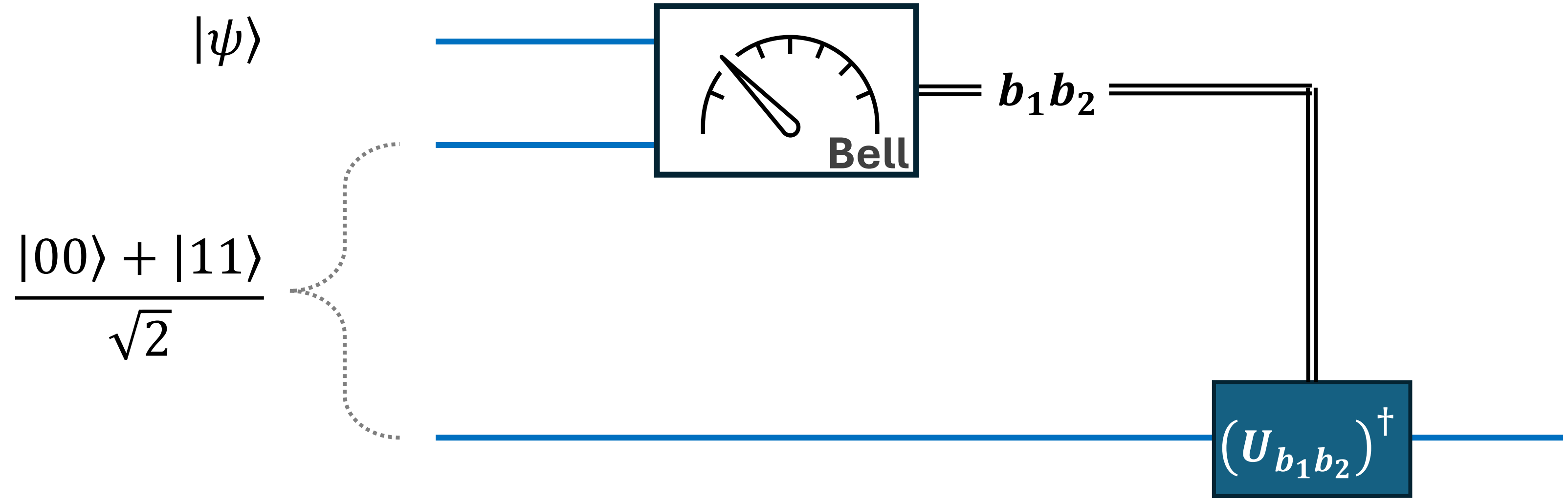
- **A more compact description (for analyzing states):**



Total states: $|\phi_0\rangle = \frac{1}{2}\sum_{b_1,b_2} |\beta_{b_1 b_2}\rangle \otimes U_{b_1 b_2}|\psi\rangle$

Quantum Teleportation

- **A more compact description (for analyzing states):**

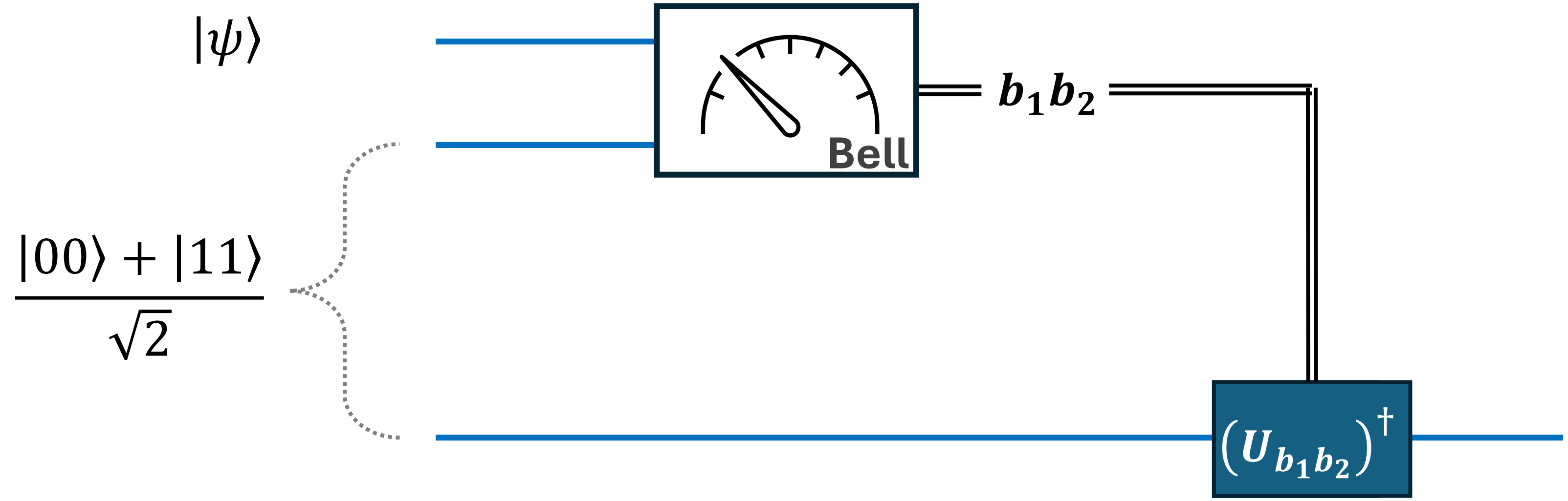


Total states:
(right after
the measurement)

$$|\phi_0'\rangle = |\beta_{b_1 b_2}\rangle \otimes U_{b_1 b_2}|\psi\rangle$$

Quantum Teleportation

- **A more compact description (for analyzing states):**

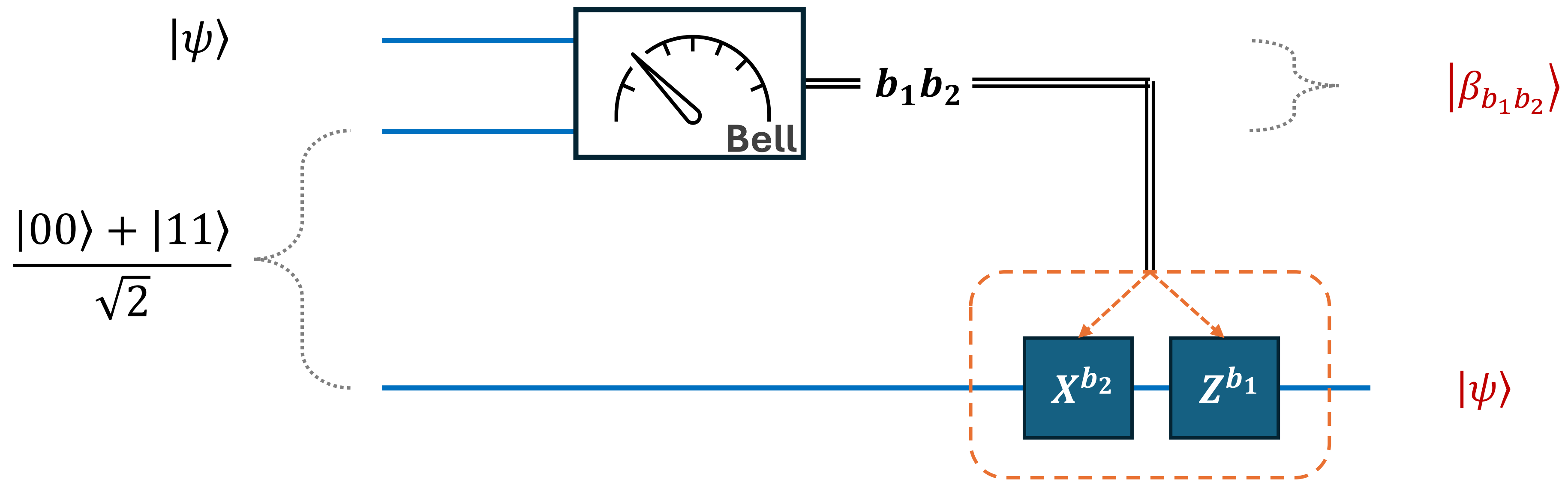


Total states:
(after applying the unitary)

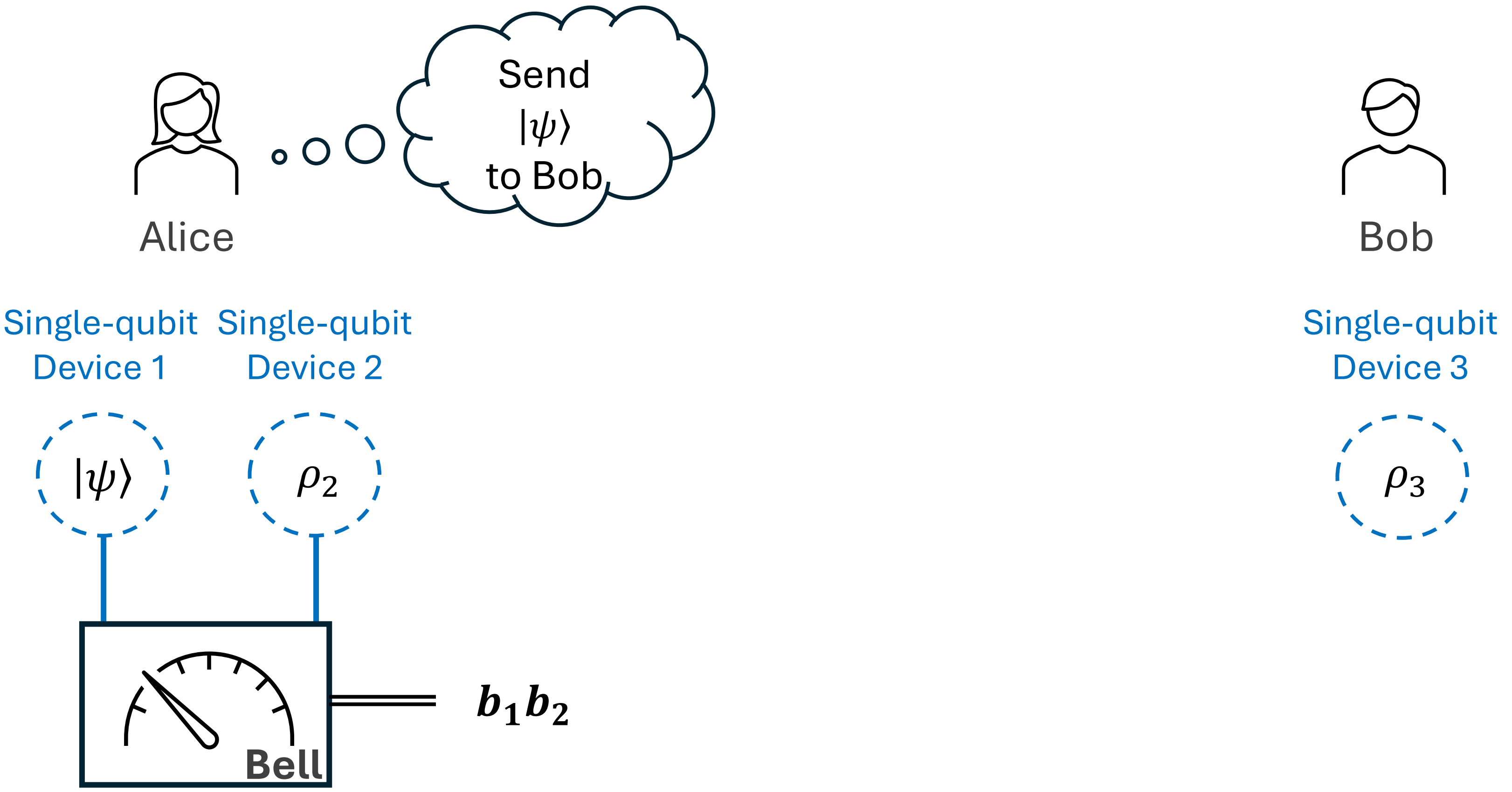
$$|\phi_1\rangle = |\beta_{b_1 b_2}\rangle \otimes |\psi\rangle$$

Quantum Teleportation

- **A more compact description (for analyzing states):**

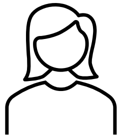


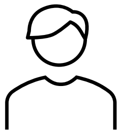
Quantum Teleportation



Total state
(before
measurement)
$|\psi\rangle \otimes |\beta_{00}\rangle$

Quantum Teleportation





Single-qubit Device 1

ρ_1'

Single-qubit Device 2

ρ_2'

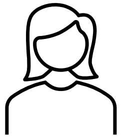
Single-qubit Device 3

ρ_3'

Total state (after measurement)

?

Quantum Teleportation

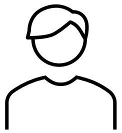


Alice

Single-qubit Device 1

Single-qubit Device 2

$|\beta_{b_1 b_2}\rangle$



Bob

Single-qubit Device 3

$\boldsymbol{U}_{b_1 b_2}|\psi\rangle$

Total state (after measurement)

$|\beta_{b_1 b_2}\rangle \otimes \boldsymbol{U}_{b_1 b_2}|\psi\rangle$

Quantum Teleportation

Alice

Single-qubit Device 1 Single-qubit Device 2

$|\beta_{b_1 b_2}\rangle$

Bob

Single-qubit Device 3

$\boldsymbol{U}_{b_1 b_2}|\psi\rangle$

Total state (after measurement)

$|\beta_{b_1 b_2}\rangle \otimes \boldsymbol{U}_{b_1 b_2}|\psi\rangle$

$\boldsymbol{b_1 b_2}$ → Send the classical bits via a classical channel

Quantum Teleportation

Alice

Single-qubit Device 1

Single-qubit Device 2

$|\beta_{b_1 b_2}\rangle$

Bob

Single-qubit Device 3

$\boldsymbol{U}_{b_1 b_2}|\psi\rangle$

$\left(\boldsymbol{U}_{b_1 b_2}\right)^\dagger$

Total state (after measurement)

$|\beta_{b_1 b_2}\rangle \otimes \boldsymbol{U}_{b_1 b_2}|\psi\rangle$

Quantum Teleportation

Alice

Single-qubit Device 1 | Single-qubit Device 2

$|\beta_{b_1 b_2}\rangle$

Bob

Single-qubit Device 3

$\boldsymbol{U}_{b_1 b_2}|\psi\rangle$

$\left(\boldsymbol{U}_{b_1 b_2}\right)^{\dagger}$

$|\psi\rangle$

Final state:

$|\beta_{b_1 b_2}\rangle \otimes |\psi\rangle$

Reference

- **[NC00]:** Sections 1.3.6 and 1.3.7
- **[KLM07]:** Chapter 5